Non-interacting electrons in disordered metal

In Boltzmann formalizm electrons are described by the distribution function in the phase space $f(\vec{p},\vec{r},t)$

This is a classical description; in quantum mechanics the operators $[\vec{r},\hat{p}] = \hbar$ therefore $\Delta x \Delta p \geq 1/2$ and one can not attripute both momentum and spatial coordinate to the particle.

We can use this approximation, provided external fields do not change significantly on the scale λ_F.

In the absence of scattering the evolution is governed by the Liuville's operator

$$\frac{df}{dt} = 0 \longrightarrow \frac{\partial f}{\partial t} + \dot{r}\nabla_r f + \dot{p}\nabla_p f = 0$$

$$\dot{r} = \vec{v} = \frac{1}{\hbar}\frac{\partial \epsilon(k)}{\partial k} \simeq \frac{\hbar \vec{k}}{m_*}$$

$$\hbar\dot{\vec{k}} = \vec{F} = (-e)\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

In addition, there is a scattering between electrons and the static disorder

$$\frac{df}{dt} = \underset{\uparrow}{I}[f] \qquad \text{collision integral}$$

$$\frac{\partial f}{\partial t} + \vec{v}\nabla_{r} f + \frac{\vec{F}}{\hbar}\nabla_{k} f = I[f]$$

Consider only the scattering between the electrons and static impurities

This process conserves the energy of electrons and is local in space.

To construct the collision integral we use scattering probability

$W_{\vec{k},\vec{k}'}$ — the probability to scatter from state $\vec{k}$ to state $\vec{k}'$

This probability is computed via scattering theory in quantum mechanics.

Within Born approximation

$$W_{k,k'} = \frac{2\pi}{\hbar}\, n_i \;\delta(\varepsilon(k)-\varepsilon(k'))\left|\langle\vec{k}|V|\vec{k}'\rangle\right|^2 =$$

impurity concentration

energy conservation

Fermi golden rule

$$= \delta(\varepsilon_k - \varepsilon_{k'})\,\omega_{k,k'}$$

The flux in the momentum space from the state $k \longrightarrow k'$

$$W_{kk'}\, f(k)\,(1-f(k'))$$

The flux in the momentum space from $k' \to k$

$$W_{k'k}\, f(k')\,(1-f(k))$$

The total flux

$$W_{k,k'}\, f(k)\,(1-f(k')) - W_{k'k}\, f(k')\,(1-f(k))$$

The collision integral

$$I[f] = \sum_{k'} \left\{ W_{k'k}\, f(k')(1-f(k)) - W_{kk'} f(k)(1-f(k')) \right\}$$

$$W_{kk'} = W_{k'k}$$

$$I[f] = -\sum_{k'} W_{kk'} \left[f(k) - f(k') \right]$$

dc electric conductivity

$$\cancel{\frac{\partial}{\partial t} f} + \cancel{\vec{v}\,\nabla_r f} - e\vec{E}\,\frac{1}{\hbar}\,\nabla_k f = -\int (dk')\, W_{kk'} \left[f(k) - f(k') \right]$$

In the absence of the external electric field the distribution is thermal

Linear respondse theory

$$f = f_0 + \delta f$$

δf linear in E correction

The left hand side of the B.E.

$$-e\vec{E}\,\frac{1}{\hbar}\nabla f \simeq -e\vec{E}\,\frac{1}{\hbar}\nabla f_0(\varepsilon_k) =$$

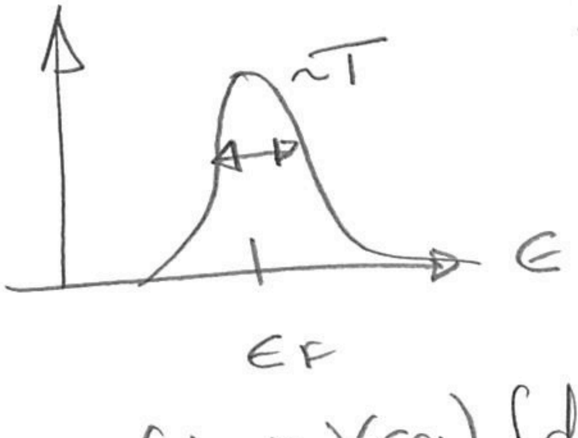
$$= -e\vec{E}\cdot\vec{v}\,\frac{\partial f_0}{\partial \epsilon_k} \qquad \text{where } \vec{v} = \frac{1}{\hbar}\nabla_k \epsilon_k = \frac{\hbar\vec{k}}{m}$$

Look for the solution of the B.E. in the following form

$$\delta f(k) = -\left(\frac{\partial f_0(\epsilon_k)}{\partial \epsilon_k}\right) F(n_{\vec{k}})$$

3D : $\vec{n}_k \equiv (\theta, \varphi)$

2D : $\vec{n}_k \equiv \varphi$



$$\int \frac{d^d\vec{k}'}{(2\pi)^d} = \int d\epsilon_{k'}\,\nu(\epsilon_{k'}) \int d\Omega_{k'}$$

$\frac{d\varphi}{2\pi}$ in 2D $\qquad \frac{\sin\theta\, d\theta\, d\varphi}{4\pi}$ in 3D

B.E.

$$-e\vec{E}\cdot\vec{v} = \nu \int d\Omega_{k'} \left[F(n_k) - F(n_{k'})\right] \omega_{kk'}$$

For 2D case $\vec{E} \parallel x$

$$-eEv\cos\varphi = \nu \int \frac{d\varphi'}{2\pi} \left[F(\varphi) - F(\varphi')\right] \omega(\varphi - \varphi')$$

The integral equation has a solution

$$F(\varphi) = -eEv\cos\varphi\,\underset{\uparrow}{\tau}$$

constat (needs to be determined)

$$\int d\varphi' (\cos\varphi - \cos\varphi') W(\varphi - \varphi') =$$

$$= \cos\varphi \int d\varphi'\, \omega(\varphi - \varphi') - \int d\varphi' \cos\varphi' W(\varphi - \varphi') =$$

$$= \cos\varphi \int d\varphi'\, \omega(\varphi - \varphi') - \int d\varphi'' \cos(\varphi'' + \varphi)\, \omega(\varphi'') =$$

$$= \cos\varphi \int d\varphi'\, \omega(\varphi') - \int d\varphi' (\cos\varphi\cos\varphi' - \underbrace{\sin\varphi\sin\varphi'}_{\text{odd func.}})\,\omega$$

$$= \cos\varphi \int d\varphi'\, \omega(\varphi') - \cos\varphi \int d\varphi'\, \omega(\varphi') =$$

$$= \cos\varphi \int d\varphi'\, \omega(\varphi')(1 - \cos\varphi)$$

$$-eEv\cos\varphi = -eEv\nu\tau\cdot\int\frac{d\varphi'}{2\pi}\omega(\varphi')(1-\cos\varphi'))\cdot\cos\varphi$$

$$1 = \nu\tau\int\frac{d\varphi'}{2\pi}\,\omega(\varphi')(1-\cos\varphi')$$

$$\boxed{\frac{1}{\tau} = \nu\int\frac{d\varphi'}{2\pi}\,\omega(\varphi')(1-\cos\varphi')}$$

$\uparrow$ transpor time (momentum relaxation time)

This is a time that typically takes for electron to randomize it's momentum.

Electric current

$$\vec{j} = \int\frac{d^dk}{(2\pi)^d}(-e)\vec{v}_k f(k)$$

2D, $\vec{E}\parallel x$

$$j_x = \int\frac{d^2k}{(2\pi)^2}(-e)v_k\cos\varphi\,\delta f(k) =$$

$$= \int d\epsilon_k\,\nu(\epsilon_k)\int\frac{d\varphi}{2\pi}(-e)v_k\cos\varphi\frac{\partial f_0}{\partial\epsilon_k}e\vec{E}v_k\cos\varphi\tau$$

$$= e^2\int d\epsilon_k\left(-\frac{\partial f_0}{\partial\epsilon_k}\right)\frac{v_k^2\tau}{2}E \overset{T\ll\epsilon_F}{\simeq} e^2\frac{\nu(\epsilon_F)}{2}\tau E = \sigma E$$

$$\sigma = e^2\nu\frac{v_F^2}{2}\tau = e^2\nu D$$

Similar for d dimension, the solution

$F(n_k) = -e\vec{E}\cdot\vec{\upsilon}\,\tau$ with transport time

$$\tau^{-1} = \nu\int d\Omega_{k'}\,\omega_{kk'}\left(1-\hat{k}\hat{k}'\right)$$

$$\Big[\int d\Omega_{k'}\,W_{kk'}F(k') = e\vec{E}\,\tau\underbrace{\int d\Omega_{k'}\,\omega_{kk'}\vec{\upsilon}'}_{\parallel\vec{k}} =$$

$$= e\vec{E}\,\hat{k}\,\tau\int d\Omega_{k'}\,\omega_{kk'}\,\vec{\upsilon}'\cdot k\Big]$$

$$\vec{J} = -e\int\frac{d^dk}{(2\pi)^d}\,\upsilon_{\vec{k}}\,f(\vec{k}) = -e\int d\epsilon_k\frac{\partial f_0}{\partial\epsilon_k}\nu(\epsilon_k)\int d\Omega\,\upsilon_k\,e\vec{E}\,\upsilon_k\tau$$

$$= \sigma\vec{E}$$

$$\sigma = e^2\int d\epsilon_k\left(-\frac{\partial f_0}{\partial\epsilon_k}\right)\nu(\epsilon_k)\frac{\upsilon_k^2}{d}\tau_k \overset{T\ll E_F}{\simeq} \frac{e^2\nu(\epsilon_F)\tau}{d}.$$

Diffusion, Einstein relation

Let us discuss the situation where is no electric field, but there is a density gradient

$n_1 - n_2 \to 0$

$\nabla n \parallel x$

$|\vec{\nabla} n| \to 0$

$$\vec{v}\nabla_r f = -\int \frac{dk'}{(2\pi)^d} W_{kk'}\left[f(k)-f(k')\right]$$

$$f = f_0(\epsilon_k - \mu(r)) + \delta f$$

$$\vec{v}\nabla_r f \simeq v\nabla_r f_0(\epsilon_k - \mu(r)) = -\vec{v}\frac{\partial f_0}{\partial \epsilon_k}\nabla\mu(r) =$$

$$= -v\cos\varphi\frac{\partial f_0}{\partial \epsilon_k}|\nabla\mu|$$

for $T=0$ $\mu = \epsilon_F$ $\nabla\mu = \frac{1}{\nu}\nabla n$

$$\delta f(k) = -\frac{\partial f_0}{\partial \epsilon_k}F(n\vec{k})$$

φ in 2D

$$-\frac{v_F}{\nu}|\nabla n|\cos\varphi = \nu\int\frac{d\varphi'}{2\pi}\left[F(\varphi)-F(\varphi')\right]\omega(\varphi-\varphi')$$

This is the same eq. we solved before with a replacement $-e\vec{E} = -\frac{1}{\nu}\nabla n$

$$\Rightarrow \quad F(\varphi) = -\frac{1}{\nu} \nabla n \, v_F \cos\varphi \, \tau$$

$$j_x = e\nu \frac{v_F^2}{2} \tau \left(-\frac{1}{\nu}\right) \nabla_x n = -eD \nabla_x n$$

$$D = \frac{v_F^2 \tau}{2} \quad \text{diffusion coefficient}$$

$\uparrow$ d in D dimensions

$$\vec{j} = -eD\nabla n$$

$$\boxed{\sigma = e^2 \nu D}$$ Einsten relation

Alternative notations

$$\sigma = e^2 \frac{\partial n}{\partial \mu} D$$

For non degenerate gas $(T \gg E_F)$

$$\frac{\partial n}{\partial \mu} = \frac{n}{k_B T}$$

For degenerate gas $T \ll E_F$

$$\frac{\partial n}{\partial \mu} = \nu(\epsilon_F)$$

ac conductivity

$$\frac{\partial}{\partial t} f - e\vec{E}(t)\frac{1}{\hbar}\nabla_k f = -\int \frac{d^d k'}{(2\pi)^d}\left[f(k) - f(k')\right] W_{kk'}$$

$E(t) = E e^{-i\omega t}$ after linearization

$$\frac{\partial f}{\partial t} - e\vec{E}\,\vec{v}_k \frac{\partial f_0}{\partial \epsilon_k} = -\int \frac{d^d k'}{(2\pi)^d} W_{kk'}\left[f_k - f_{k'}\right]$$

$$\frac{\partial}{\partial t} \to -i\omega \qquad \delta f_k(t) = \delta f_k e^{-i\omega t}$$

$$\delta f(k) = -\frac{\partial f_0}{\partial \epsilon} F(\varphi)$$

$$\nu \int \frac{d\varphi'}{2\pi}\left[F(\varphi) - F(\varphi')\right] \omega(\varphi - \varphi') - i\omega F(\varphi) = -eEv_F \cos\varphi$$

$$F(\varphi) = -eEv_F \cos\varphi\, \tilde{\tau}$$

$$(\tau^{-1} - i\omega)\tilde{\tau} = 1 \qquad \tilde{\tau} = \frac{\tau}{1 - i\omega\tau}$$

$$\boxed{\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}} \qquad \sigma_0 = e^2 \nu v_F^2 \frac{\tau}{2} = e^2 \nu D$$

In time representation

$$\frac{\partial F(\varphi,t)}{\partial t} + \nu \int \frac{d\varphi'}{2\pi}\left[F(\varphi,t)-F(\varphi',t)\right]\omega(\varphi-\varphi') = $$
$$= -eE(t)v_F\cos\varphi$$

$$F(\varphi,t) = -ev_F\cos\varphi\,\tilde{F}(t)$$

$$\frac{\partial \tilde{F}(t)}{\partial t} + \frac{1}{\tau}\tilde{F}(t) = E(t)$$

$$\tilde{F}(t) = \int_{-\infty}^{t} dt'\, e^{-\frac{t-t'}{\tau}}\, E(t')$$

$$j(t) = -e\nu v_F \int d\varphi \cos\varphi\; F(\varphi,t) =$$
$$= \frac{e^2\nu v_F^2}{2}\int_{-\infty}^{t} dt'\, \underbrace{e^{-\frac{(t-t')}{\tau}}}_{\text{memory-function}}\, E(t') =$$

$$= \int_{-\infty}^{t} \sigma(t-t')E(t')\,dt'$$

$$\sigma(t) = \frac{e^2\nu v_F^2}{2}e^{-t/\tau}$$

$$\sigma(\omega) = \int_0^{\infty} dt\, e^{i\omega t}\,\sigma(t)$$

Classical kubo formula

$$\sigma(t) = e^2 \nu \langle v_x(0) v_x(t) \rangle$$

averaging over the ensemble

To demonstrate kubo formula, consider the fermion with the velocity $\vec{v}_0$ at the time $t=0$

$$F(\varphi, t=0) = \delta(\varphi - \varphi_0) \qquad (*)$$

The homogeneous eq. reads

$$\frac{\partial F(\varphi,t)}{\partial t} + \nu \int \frac{d\varphi'}{2\pi} \left[F(\varphi,t) - F(\varphi',t) \right] w(\varphi-\varphi') = 0$$

with the initial condition $(*)$

Let us compute

$$\left\langle v_x(t) \right\rangle = v_F \int F(\varphi,t) \cos\varphi \frac{d\varphi}{2\pi}$$

Fourier transfor

$$F(\varphi,t) = \frac{1}{2\pi} \sum_n \left[\cos n\varphi \, F_n^{(e)}(t) + \sin n\varphi \, F_n^{o}(t) \right]$$

$$\frac{\partial F_1^{(e)}(t)}{\partial t} + \frac{1}{\tau} F_1^{(e)}(t) = 0 \Rightarrow F_1^{(e)}(t) = F_1^{(e)}(0) e^{-t/\tau}$$

$$F_1^{(e)}(0) = \cos\varphi_0$$

$$\langle v_x(t)\rangle = v_F \cos\varphi_0 \, e^{-t/\tau}$$

$$\langle v_x(0)\, v_x(t)\rangle = v_F^2 \int \cos^2\varphi_0 \, e^{-t/\tau} \frac{d\varphi_0}{2\pi} =$$

$$= \frac{v_F^2}{2} e^{-t/\tau}$$

$$\sigma(t) = e^2 \nu \langle v_x(0)\, v_x(t)\rangle \, \theta(t)$$

τ-approximation

Instead of solving the integral equation one can replace the collision integral

$$I[f] \to -\frac{f(k) - f_0(k)}{\tau_{tr}}$$

replacing the matrix that represent the linearized coll. integral $I[f] = \sum_{k'} I_{kk'} \delta f_{k'}$ by its diagonal part. In many cases this approximation is enough to estimate scales in the problem, and for very simple models (constant fields, parabolic spectrum etc.) gives the correct result.

Magnetoconductivity

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla_r f - \frac{e}{c}\vec{v}\times\vec{B}\,\frac{1}{\hbar}\nabla_k f - eE\nabla_k f = I[f]$$

2D $\quad \vec{B} \parallel z$

$$\vec{v}\times\vec{B}\,\frac{1}{\hbar}\nabla_k f = (\vec{v}\times\vec{B})\,\vec{v}\,\frac{\partial f}{\partial \epsilon_k} - \frac{1}{m} B \frac{\partial f}{\partial \varphi_k}$$

$$f = f_0 + \delta f$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\nabla_r - \frac{eB}{mc}\frac{\partial}{\partial \varphi} - \hat{I}\right)\delta f = eE\cos\varphi\frac{\partial f_0}{\partial \epsilon}$$

$$\delta f = \left(-\frac{\partial f_0}{\partial \epsilon_k}\right) F(\varphi)$$

Consider the static electric field $\vec{E}$

$$\left(\omega_c\frac{\partial}{\partial \varphi} - \hat{I}\right) F = -ev\cos\varphi$$

$$\omega_c = \frac{eB}{mc}$$

$$I[\cos\varphi] = -\frac{1}{\tau}\cos\varphi$$

$$I[\sin\varphi] = -\frac{1}{\tau}\sin\varphi$$

$$\left(\omega_c \frac{d}{d\varphi} + \frac{1}{\tau}\right) F = -eE\, v \cos\varphi$$

$$F = -\frac{eE v_F \tau}{1+\omega_c^2 \tau_c^2}\left(\cos\varphi + \omega_c\tau \sin\varphi\right)$$

$$j_x = \nu \int \frac{d\varphi}{2\pi} (-e)\, v_F \cos\varphi\, F = \frac{e^2 \nu v_F^2}{2} \frac{\tau}{1+\omega_c^2\tau^2} E$$

$$j_y = \nu \int \frac{d\varphi}{2\pi} (-e)\, v_F \sin\varphi\, F = e^2 \nu \frac{v_F^2}{2} \frac{\omega_c \tau^2}{1+\omega_c\tau} E$$

$$\vec{j} = \sigma \vec{E} \; ; \qquad \hat{\sigma} = \frac{\sigma_0}{1+\omega_c^2\tau^2}\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix}$$

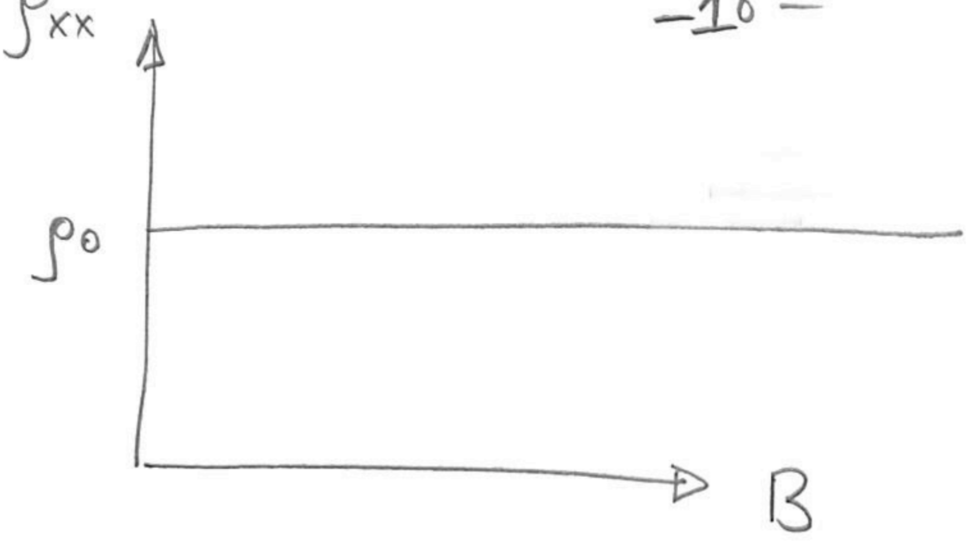
$$\sigma_0 = e^2 \nu \frac{v_F^2 \tau}{2} = \frac{n e^2 \tau}{m}$$

conductivity tensor

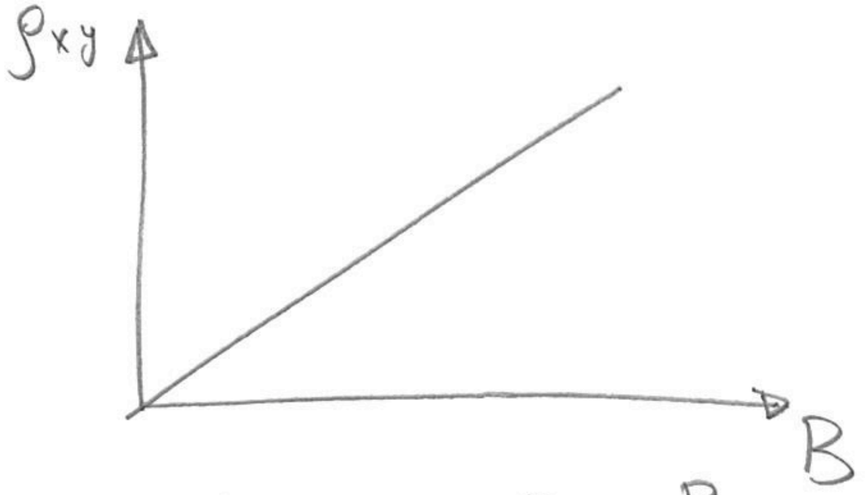
Resistivity tensor

$$\hat{\rho} = \hat{\sigma}^{-1} = \rho_0 \begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}$$

$$\rho_0 = \sigma_0^{-1}$$



independent of B



$$\rho_{xy} \equiv R_H = \frac{B}{nec}$$ independent of τ

Note that Einstein relation

$\hat{\sigma} = e^2 \nu \hat{D}$ holds for finite magnetic field, e-e interaction

$\vec{j} = -eD\nabla n$ diffusion current

$$\hat{D}_{ij} = \int_0^\infty dt \, \langle \vec{v}_i(t) \, \hat{v}_j(0) \rangle$$

matrix of diffusion coefficients

Appendix:

Diffusion approximation fo non-interacting electrons

$$\frac{\partial f}{\partial t} + v_p \nabla f + I[f] = 0$$

Sum over p

$$\frac{\partial}{\partial t} \sum_p f(p,r,t) + \nabla_r \left(\sum_p v_p f \right) = 0$$

$$\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial r} \left(j(r) \right) = 0$$

$$j = \sum_p v_p f$$

Solve the problem $t=0$ $\rho = \rho_0 + \delta\rho(x)$

How the density of electrons spreads in the non-interacting disordered metal

Assume that non-equilibrium is induced in x direction.

$$f(p,r,t) = f_0(\epsilon_p, r, t) + \vec{f_1}(\epsilon_p, r, t) \cdot \hat{p}$$

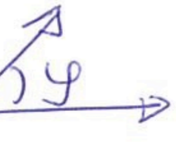
$$f_1 \ll f_0$$

$$I[f_p] = \sum_{p'} W_{p,p'} \{ f_p (1 - f_{p'}) - f_{p'} (1 - f_p) \}$$

$$= \sum_{p'} (f_p - f_{p'}) W_{p,p'}$$

Let us focus on $d=2$

Non equlibrium direction and p



$$f(p,r,t) = f_0 + f_1 \cos\varphi + f_2 \cos 2\varphi$$

$$I[f] = \frac{f_1 \cos\varphi}{\tau_{tr}}$$

$$I[f_0] = 0$$

Substitute this solution to the B.e.

$$\frac{\partial}{\partial t}(f_0 + f_1 \cos\varphi) + \vec{v}_p \cdot \nabla(f_0 + f_1 \cos\varphi) + \frac{f_1 \cos\varphi}{\tau_{tr}} = 0$$

$$v_p \cdot \nabla f_0 = \frac{\vec{p}}{m} \cdot \nabla f_0 = \frac{p}{m} \cos\varphi \left|(\nabla f_0)\right|$$

Harmonics $\cos\varphi$

$$\frac{\partial}{\partial t} f_1 \cos\varphi + \frac{p}{m}|\nabla f_0| \cos\varphi + \frac{f_1 \cos\varphi}{\tau_{tr}} = 0$$

$$i\omega f_1 + \frac{p}{m}|\nabla f_0| + \frac{f_1}{\tau_{tr}} = 0$$

$$f_1\left(i\omega + \frac{1}{\tau_{tr}}\right) = -\frac{p}{m}|\nabla f_0|$$

$$\boxed{f_1 = -\frac{p}{m}\frac{|\nabla f_0|}{i\omega + 1/\tau_{tr}} = -\frac{p}{m}\frac{\tau_{tr}|\nabla f_0|}{1 + i\omega\tau_{tr}}}$$

$\omega\tau \ll 1$

this canbe neglected

$$\frac{\partial}{\partial t} f_0 + \vec{v}_F \cdot \nabla f_1 \cos\varphi = 0$$

$$\frac{\partial}{\partial t} f_0 + \left|\frac{\vec{p}}{m}\right| \cos^2\varphi \left|\nabla f_1\right| = 0$$

$$\frac{\partial}{\partial t} f_0 + \cos^2\varphi \left|\frac{p}{m}\right|^2 \left(-\tau_{tr}\right) \nabla^2 f_0 = 0$$

$$\frac{\partial}{\partial t} f_0 - v_F^2 \tau_{tr} \cos^2\varphi \, \nabla^2 f_0 = 0$$

This is not valid, since f_0 does not depend on φ. But let us average over φ, i.e $\cos^2\varphi = \frac{1}{2} + \underbrace{\frac{\cos 2\varphi}{2}}_{\text{neglect}}$

$$\frac{\partial}{\partial t} f_0 - \frac{v_F^2}{2} \tau_{tr} \nabla^2 f_0 = 0$$

$$\boxed{\frac{\partial}{\partial t} f_0 - D \nabla^2 f_0 = 0}$$ diffusion equation.

We made an approximanion in replacing $\cos^2\varphi$ (i.e. second harmonic term) by its average value. We can correct it by accounting for $\cos 2\varphi$ terms in the expansion, and generate an erron in higher harmonics and so on.

But now we have an equation for t-dependent solution of B.e.

Integrate it over p

$$\frac{\partial}{\partial t}\rho - D\nabla^2\rho = 0$$

we get the ordinary diffusion equation.
Given the continuity equation we find
that in non interacting metal

$$j = -D\nabla\rho$$

at scales longer $l = v_F \tau_{tr}$

the spread is diffusive.
Similar derivation yield the same
result at any dimension with $D = \frac{v_F^2 \tau_{tr}}{d}$