Electron-electron scattering   (p. 132 Gant.-Levinson)

Part 1

$$e\vec{E} + e\vec{B} \times \vec{v}_p = e\vec{E} + e\vec{v}_p \times \vec{B}$$

$$\frac{\partial f}{\partial t} + \vec{v}_p \cdot \nabla_r f + \vec{F} \cdot \nabla_p f = I[f]_{ee}$$

$$I_{ee}[f] = I_{out}[f] + I_{in}[f]$$

$$I_{out}[f] = -\sum_{k_1, k', k_i'} W_{k, k_1}^{k' k_i'} \, f_k \, f_{k_1} (1 - f_{k'})(1 - f_{k_1})$$

$$I_{in}[f] = \sum_{k, k_1}^{'} W_{k, k_1}^{k' k_i'} \, (1 - f_k)(1 - f_{k_1}) f_{k'} f_{k_i'}$$

To compute the matrix element W we
use Fermi golden rule

$$W_{i \to f} = \frac{2\pi}{\hbar} \left| \langle f | T | i \rangle \right|^2 \delta(\epsilon_i - \epsilon_f)$$

Computation of the matrix element.

In real materials the interaction is Coulomb (bare)

& $U(r) = \dfrac{e^2}{r}$

Properties of the Coulomb interaction:

1) It is spin independent. However the
matrix element of two electron interaction
depends on the total spin.

Focus on two electrons. In the absence of interaction the wave function is given by slater determinant (for spinless electrons)

$$\Psi_{1,2}(r_1,r_2)=\frac{1}{\sqrt{2}}\left[\Psi_1(r_1)\Psi_2(r_2)\mp\Psi_1(r_2)\Psi_2(r_1)\right]$$

For spinfull electron it corresponds to $+$ for $S=0$ (singlet state), ~~For triplet~~ ~~S=1~~

and $-$ for $S=1$ (triplet).

Discuss this case (interaction in the singlet channel). triplet channel is obtained by replacement $+$ by $-$.

$$M_{1,2\to 1',2'}=\int d^d r_1\, d^d r_2\ \Psi^*_{1,2}(r_1,r_2)\, U(r_1-r_2)\, \Psi_{1',2'}(r_1,r_2)=$$

$$=\int d^d r_1\, d^d r_2\ \frac{1}{2}\left(\Psi^*_1(r_1)\Psi^*_2(r_2)+\Psi^*_1(r_2)\Psi^*_2(r_1)\right)V(r_1-r_2)$$
$$\left(\Psi_{1'}(r_1)\Psi_{2'}(r_2)+\Psi_{1'}(r_2)\Psi_{2'}(r_1)\right)=$$

$$=\frac{1}{2}\int d^d r_1\, d^d r_2\left[\Psi^*_1(r_1)\Psi^*_2(r_2)\ \Psi_{1'}(r_1)\Psi_{2'}(r_2)+\right.$$
$$\Psi^*_1(r_1)\Psi^*_2(r_2)\ \Psi_{1'}(r_2)\Psi_{2'}(r_1)+$$
$$\Psi^*_1(r_2)\Psi^*_2(r_1)\ \Psi_{1'}(r_1)\Psi_{2'}(r_2)+$$
$$\left.\Psi^*_1(r_2)\Psi^*_2(r_1)\ \Psi_{1'}(r_2)\Psi_{2'}(r_1)\right]V(r_1-r_2)=$$

$$=\frac{1}{2}\int d^d r_1\, d^d r_2\left[\Psi^*_1(r_1)\Psi^*_2(r_2)\Psi_{1'}(r_1)\Psi_{2'}(r_2)+\right.$$
$$\left.+\Psi^*_1(r_1)\Psi^*_2(r_2)\Psi_{1'}(r_2)\Psi_{2'}(r_1)+(r_1\leftrightarrow r_2)\right]V(r_1-r_2)$$

symmetric under $r_1\leftrightarrow r_2$

$$= \int d^d r_1 \, d^d r_2 \left[\Psi_1^*(r_1) \Psi_2^*(r_2) \Psi_{1'}(r_1) \Psi_{2'}(r_2) + \right.$$

direct ↙

$$\left. + \Psi_1^*(r_1) \Psi_2^*(r_2) \Psi_{1'}(r_2) \Psi_{2'}(r_1) \right] V(r_1 - r_2)$$

↑ exchange

For the electrons in the crystal

$\Psi_1(r)$ is a Bloch function, i.e.

$$\Psi_{k_1}(r) = e^{i \vec{k}_1 \vec{r}} u_{k_1}(\vec{r})$$

ignore the crystal for a moment and study electrons in the "vacuum".

$$M_{1,2 \to 1',2'} = M_{k_1, k_2 \to k_1', k_2'} =$$

$$= \int d^d r_1 \, d^d r_2 \left[e^{-i k_1 r_1 - i k_2 r_2 + i k_1' r_1 + i k_2' r_2} \right. +$$

$$\left. + e^{-i k_1 r_1 - i k_2 r_2 + i k_1' r_2 + i k_2' r_1} \right] U(r_1 - r_2) =$$

$$= \int d^d r_1 \, d^d r_2 \left[e^{i r_1 (k_1' - k_1) + i r_2 (k_2' - k_2)} + e^{i r_1 (k_2' - k_1) + i r_2 (k_1' - k_2)} \right]$$

$$U(r_1 - r_2)$$

Define $\quad \vec{r}_- = \vec{r}_1 - \vec{r}_2$

$$\vec{R}_+ = \frac{\vec{r}_1 + \vec{r}_2}{2}$$

$$r_1 = \frac{r_-}{2} + R_+$$

$$r_2 = R_+ + r_-/2$$

$$M_{k_1, k_2 \to k_1' k_2'} = \int d^d r_- \, d^d R_+ \left\{ e^{i\left(R_+ + \frac{r_-}{2}\right)(k_1' - k_1) + i\left(R_+ - \frac{r_-}{2}\right)(k_2' - k_2)} \right.$$

$$\left. + e^{i\left(R_+ + \frac{r_-}{2}\right)(k_2' - k_1) + i\left(R_+ - \frac{r_-}{2}\right)(k_1' - k_2)} \right\} U(r_-) =$$

$$= \int \frac{d^d R_+}{\cancel{d R_+}} \int d^d r_- \left\{ \delta(k_1' - k_1 + k_2' - k_2) \, e^{i\frac{r_-}{2}(k_1' - k_1 - k_2' + k_2)} \right. +$$

$$\left. + \delta(k_2' + k_1' - k_1 - k_2) \, e^{i\frac{r_-}{2}(k_2' - k_1 - k_1' + k_2)} U(r_-) \right\} =$$

$$= \delta(k_1' + k_2' - k_1 - k_2) \int d^d r_- \left[e^{i(k_2 - k_2')r_-} + e^{i(k_2' - k_1)r_-} \right] U(r_-)$$

$$= \delta(k_1' + k_2' - k_1 - k_2) \left[U(k_2 - k_2') + U(k_1 - k_2') \right] =$$

$$= \delta(k_1' + k_2' - k_1 - k_2) \left[U(k_1 - k_1') + U(k_1 - k_2') \right] =$$

$$= \delta(k_1' + k_2' - k_1 - k_2) \left[U(k_1 - k_1') \pm U(k_1' - k_2) \right]$$

$$\left(\begin{array}{l} + \text{ for } S = 0 \\ - \text{ for } S = 1 \end{array} \right)$$

Here the Fourier component of bare interaction

$$U(q) = \int d^d r \, e^{iqr} U(r) = \frac{4\pi e^2}{q^2}$$

$$U(q) \xrightarrow[q \to 0]{} \infty \qquad \text{therefore the probability}$$

$$W_{k_1 \to k_1'} \text{ diverges for small}$$

angle scattering.

Scattering time

$$\frac{1}{\tau_{k}} \propto \int (d^{d}k') \, W_{k \to k'} =$$

$$\int (d^{d}k') \, |M_{k, k_{2}, k', k_{2}'}|^{2} \, \delta(\epsilon_{k} + \epsilon_{k_{2}} - \epsilon_{k'} - \epsilon_{k_{2}'})$$

$$= \int (d^{d}k') \, \delta(k' + k_{2}' - k - k_{2}) \, \delta(\epsilon_{k} + \epsilon_{k_{2}} - \epsilon_{k'} - \epsilon_{k_{2}'})$$

$$\frac{1}{(k-k')^{4}} \qquad (d^{d}k_{2})(d^{d}k_{2}') =$$

$$= \int (d^{d}k_{2})(d^{d}k') \, \delta\left(\frac{k^{2}}{2m} + \frac{k_{2}^{2}}{2m} - \frac{k'^{2}}{2m} - \frac{(k'-k-k_{2})^{2}}{2m} \right)$$

$$\frac{e^{4}}{(k-k')^{4}} =$$

$$= 2m \int (d^{d}k')(d^{d}k_{2}) \, \delta\left(k^{2} + k_{2}^{2} - k'^{2} - (k'^{2} + k^{2} + k_{2}^{2} - 2k'k - 2k'k_{2} + 2kk_{2}) \right)$$

$$= 2m \int (d^{d}k') \, \delta\left(\underbrace{-2k'^{2} + 2kk' + 2k'k_{2} - 2kk_{2}}_{} \right) \frac{e^{4}}{(k-k')^{4}} =$$

$$\vec{k}_{2}(\vec{k} + \vec{k}') = k'^{2} - \vec{k} \vec{k}'$$

$$2\vec{k}_{2} \cdot \vec{k} \simeq \boxed{(\vec{k}' - \vec{k})\vec{k}} \qquad \qquad \vec{k} \simeq \vec{k}'$$

$$\vec{k}' \cdot (\vec{k}' - \vec{k})$$

$$2 k_{2} \cdot k \cos\theta = \vec{k}'(\vec{k}' - \vec{k}) \simeq \vec{k} \cdot (\vec{k}' - \vec{k})$$

$$\frac{1}{\tau_R} \simeq 2m \frac{1}{2} \int (d^d k')(d^d k_2) \frac{\delta(-k'^2 - \vec{k}\vec{k}_2 + \vec{k}\vec{k}' + \vec{k}'\vec{k}_2)}{(\vec{k} - \vec{k}')^4}$$

$$k_2(\vec{k}' - \vec{k}) + \vec{k}\vec{k}' - k'^2 = 0$$

$$k_2(k' - k) = k'^2 - \vec{k}\vec{k}'$$

$$\vec{k}_2(\vec{k}' - \vec{k}) = k'(\vec{k}' - \vec{k})$$

$$\vec{k}_2 \quad \vec{q} \equiv \vec{k}' - \vec{k}$$

$$K_T = \sqrt{mT}$$

$$\frac{1}{\tau_R} \simeq m \int (d^d q) \frac{e^4}{q^4} \int_0^{K_T} dk_2\, k_2^2 \int_0^{\pi} \sin\theta\, d\theta$$
$$\delta(k_2 \cos\theta\, q - q(k+q))$$

$$\simeq m \int (d^d q) \frac{e^4}{q^4} \int_0^{K_T} dk_2\, k_2^2 \int_{-1}^{1} dx\, \delta(k_2 q x - q(k+q)) =$$

$$\simeq m \int d^d q\, \frac{e^4}{q^4} \int_0^{K_T} dk_2\, \frac{1}{k_2 q}\, k_2^2 \simeq$$

$$\simeq m \int \frac{d^d q}{q^5}\, k_T^2 = m k_T^2\, q^{-2} e^4 \sim$$

$$\simeq m \int_{q_{min}} \frac{d^d q}{q^5}\, k_T^2$$

$$= \underbrace{m e^4}_{\text{Bohr energy}} k_T^2\, q_{min}^{-2}$$

diverges as $q_{min} \to 0$

Scattering time diverges as q^{-2}! -7-

To estimate transport time

$$\vec{q} = \Delta\vec{k} = \vec{k}' - \vec{k}$$

$$\Delta k_\parallel = q\cos(\pi/2 - \theta) = q\sin\theta \simeq q\theta \sim k\theta^2$$
$$q \sim k\theta$$

| parallel to k

$$\Delta k_\perp = q \sim k\theta$$
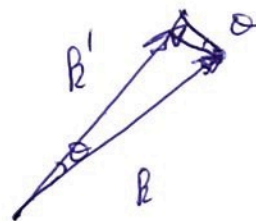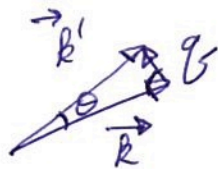
therefore $(1 - \cos\theta) \sim \theta^2 \sim q^2$ and

$$1/\tau_{tr}(k) \sim \int \frac{dq}{q^5} \cdot q^2 \simeq \ln q = \ln\theta_{min}$$

Coulomb logarithm.

To cure this divergence one needs to take into account a screening between the electrons, that sets $\theta_{min} \simeq \dfrac{1}{k\lambda_D}$

Debye screening length.

In terms of θ_{min}

$q_{min} \sim k\theta_{min} \sim 1/r_D$

$$\frac{1}{\tau(k)} \sim m e^4 k_F \theta_{min}^{-2}$$

Debye screening length.

We see that long range interaction forces to incorporate screening. Let us try to understand it, first on a qualitative level.

At equilibrium, in the absence of inhomoginuities metal is electro neutral system, with charge of electrons conpensating the charge of positive ions.
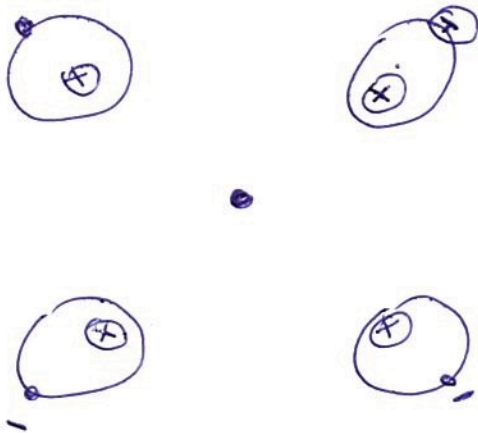

$$\rho_{electrons} = Const$$
$$= -\rho_{ions}$$
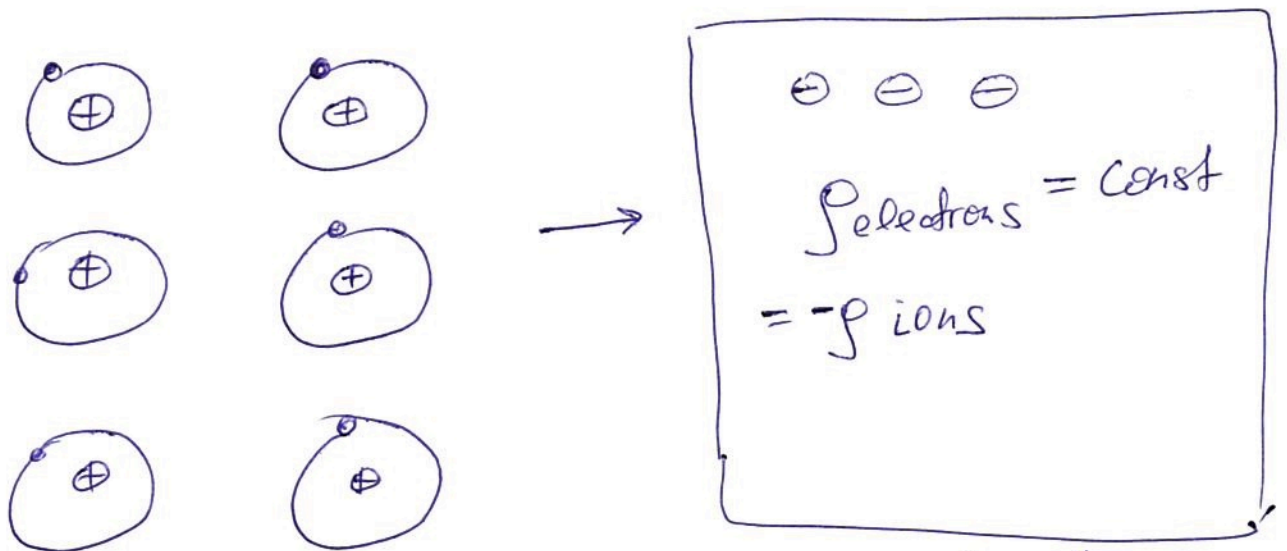
Consider ions as a background with a constant density

Add an extra electron to the system.

How does the system respond?

In dialectic, where electrons are bound to atoms the displacement is such

that creates a local field that is opposite to the field of an added charge.

This replaces an original Coulomb potential $U(g) = \dfrac{4\pi e^2}{g^2} \longrightarrow \dfrac{4\pi e^2}{\epsilon g^2}$

dialectric constant

In the metal electrons are not confined to the atoms, so they have a freedom to move and screen the additional charge completely.

To analize it, let us look at the Boltzmann eq. at equilibrium and add to it Poisson eq.

$$\text{div } \vec{E} = 4\pi \rho(\vec{r})$$

$$\rho(\vec{r}) = e\int(d^3p)\, f(\vec{p},\vec{r})$$

$$\vec{E} = -\not{\ }\nabla\varphi$$

$$\vec{v}_p\nabla_r f + e\vec{E}\cdot\nabla_p f(\vec{p},r) = 0$$

Look for solution $f(p,r) = f_{FD}\left(\dfrac{\varepsilon_p + e\varphi}{T}\right) =$

$$= \frac{1}{e^{\frac{\varepsilon_p + e\varphi}{T}} + 1}$$

$$\nabla_p f = \frac{\partial f_{FD}}{\partial\varepsilon} \qquad \nabla_p\frac{\varepsilon_p}{T} = \frac{\vec{v}_p}{T}\frac{\partial f_{FD}}{\partial\varepsilon}$$

$$\nabla_r f = \frac{\partial f_{FD}}{\partial\varepsilon} \qquad \nabla_r\frac{e\varphi}{T} = -\frac{e\vec{E}}{T}\frac{\partial f}{\partial\varepsilon}$$

$$\vec{v}_p\nabla_r f + e\vec{E}\nabla_p f = -\frac{e\vec{E}\cdot\vec{v}_p}{T}\frac{\partial f}{\partial\varepsilon} + \frac{e\vec{E}\vec{v}_p}{T}\frac{\partial f_{FD}}{\partial\varepsilon} = 0$$

Therefore the solution works!

The corresponding charge density

$$\rho(r) = e \int (d^3p) \frac{1}{e^{\frac{\epsilon_p + e\varphi}{T}} + 1} \simeq e \int d^3p\, e^{-\frac{\epsilon_p + e\varphi}{T}}$$

$$\delta\rho(r) = e \int d^3p \left[e^{-\frac{(\epsilon_p + e\varphi)}{T}} - e^{-\frac{\epsilon_p}{T}} \right] \simeq -\frac{e^2\varphi}{T} \underbrace{\int d^3p\, e^{-\frac{\epsilon_p}{T}}}_{n(T)} =$$

$$= -\frac{e^2\varphi}{T} n$$

density of electrons

~~$\triangle\varphi = $~~

Therefore $\quad \text{div } E = 4\pi\rho$

$$// $$

$$+\triangle\varphi = \frac{4\pi e^2 n}{T}\varphi$$

$$\lambda_D^{-2} = \frac{4\pi e^2 n}{T} \qquad \boxed{\lambda_D = \sqrt{\frac{T}{4\pi e^2 n}}}$$

Debye-Hückel screening length.
for large temperature.

for $T \ll \mu$

$$\rho(r) = +e \int (d^3p) \frac{\partial n}{\partial\epsilon} \cdot e\varphi = e^2 \varphi \int (d^3p) \frac{\partial n}{\partial\epsilon} =$$

$$= e^2 \varphi \nu \int d\epsilon \frac{\partial n}{\partial\epsilon} = e^2 \varphi \nu = -e^2 \nu\, \varphi$$

$$+\triangle\varphi = 4\pi e^2 \nu\, \varphi$$

$$\boxed{\lambda_D^{-2} = 4\pi e^2 \nu}$$

This equation has a spherical solutions

$$\varphi(z) = A \frac{e^{-r/\lambda_D}}{r}$$

If electric charge Q is placed at the origin

$$\Delta \varphi = -4\pi \rho$$

integrate around origin

$$\Delta \frac{1}{r} = \delta(r)(-4\pi)$$

$$\oiint \int d^3r \Delta \left(\frac{A}{r}\right) = -4\pi A \int d^3r \Delta\left(\frac{1}{r}\right) = -4\pi Q$$

$$A = Q$$

Check

$$\int \left(\Delta \frac{1}{r}\right) d^3V =$$

$$= \int_V \operatorname{div}\left(\nabla \frac{1}{r}\right) d^3V =$$

$$= \int_V \operatorname{div}\left(\frac{-\hat{r}}{r^2}\right) d^3V =$$

$$= \int_S \frac{-1}{r^2} d^2S =$$

$$= \frac{4\pi r^2}{r^2} = 4\pi$$

Therefore if the charge Q is placed at the origin the electrostatic potential that is formed due to the screening by electrons

$$\boxed{\varphi(z) = \frac{Q}{r} e^{-r/\lambda_D}}$$

Note that for $r > \lambda_D$ the potential decays exponentially, as if interaction is short range.