(Part 2)

Electron-electron scattering - Fermi gas

coll. integral

$$I_{ee}[f] = I_{out}[f] + I_{in}[f]$$

$$I_{out} = -\sum_{k_1, k_1', k'}' W_{k, k_1}^{k', k_1'} f(k) f(k_1)(1-f_{k'})(1-f_{k_1'})$$

$$I_{in} = \sum' W_{k, k_1}^{k' k_1'} (1-f_k)(1-f_{k_1}) f_{k'} f_{k_1'}$$

<u>Linearization of the collision integral</u>

$$f(k) = f_0 + \delta f$$

$$\begin{aligned} I_{out} = -\sum' W_{k,k_1}^{k',k_1'} \Big\{ & \delta f_k \, f_0(k_1)(1-f_{0k'})(1-f_{0k_1'}) + \\ & + f_0(k)\, \delta f(k_1)(1-f_{0k'})(1-f_0(k_1')) - \\ & - f(k) f(k_1)(1-f_{k_1}) \delta f(k') - \\ & - f(k) f(k_1)(1-f_{k'}) \delta f(k_1') \Big\} \end{aligned}$$

$$\begin{aligned} I_{in} = \sum' W_{k,k_1}^{k',k_1'} \Big\{ & -\delta f_k (1-f_{k_1}) f_{k'} f_{k_1'} - \delta f_{k_1} (1-f_k) f_{k'} f_{k_1'} + \\ & + \delta f_{k'} (1-f_k)(1-f_{k_1}) f_{k_1'} + \delta f_{k_1'} (1-f_k)(1-f_{k_1}) f_{k'} \Big\} \end{aligned}$$

Local part in the k-space

$$I[\delta f_k] = -\frac{\delta f(k)}{\tau_k} + \sum_{k_1} \underbrace{\delta f(k_1)\, I_{k_1,k}}$$

non local part

The diagonal part:

$$\frac{1}{\tau_k} = \sum_{k_1,k',k_1'} \Big\{ f(k_1)(1-f(k'))(1-f(k_1')) + (1-f(k_1))\, f_{k'}\, f_{k_1'} \Big\}\, W_{k k_1}^{k' k_1'} =$$

On the level of F.G.R

$$= \int (d^3k_1)(d^3k')(d^3k_1')\ \frac{2\pi}{\hbar}\, \delta(\epsilon_k + \epsilon_{k_1} - \epsilon_{k'} - \epsilon_{k_1'}) \cdot$$

$$\cdot \left| M_{k k_1}^{k' k_1'} \right|^2 \Big\{ f_{k_1}(1-f_{k'})(1-f_{k_1'}) + (1-f_{k_1})\, f_{k'}\, f_{k_1'} \Big\}$$

Matrix element

$$M_{k,k_1}^{k' k_1'} = \left(U_{k-k'} \pm U_{p-k'} \right) \times \delta(k+k_1-k'-k_1'),$$ where

$$U(q) = 4\pi e^2 \frac{\lambda^2(q)}{1+q^2\lambda^2(q)}$$ screened Coulomb potential.

For $q \to 0$ (at $T \to 0$)

$$U(q) \to 4\pi e^2 \lambda^2 = \frac{4\pi e^2}{4\pi e^2 \nu_3} \simeq \frac{1}{\nu_3} \doteq U_0$$

effective constant of interaction.

Note the electron charge has gone.

$$1/\tau_k \cong \int (d^3k_1)(d^3k')(d^3k_1') \frac{2\pi}{\hbar}$$

$$\delta(k+k_1-k'-k_1')\,\delta(\epsilon_k+\epsilon_{k_1}-\epsilon_{k'}-\epsilon_{k_1'})\,\cdot$$

$$U_0^2\left[f_{k_1}(1-f_{k'})(1-f_{k_1'})+(1-f_{k_1})f_{k'}f_{k_1'}\right]=$$

equilibrium distribution function F.D.

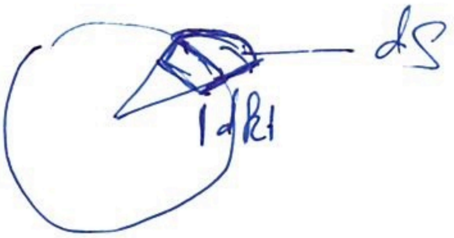
$$=\frac{2\pi}{\hbar}U_0^2\int d^3k_1\, d^3k'\,\delta(\epsilon_k+\epsilon_{k_1}-\epsilon_{k'}-\epsilon_{k+k_1-k'})$$

$$\left[f(\epsilon_{k_1})(1-f(\epsilon_{k'}))(1-f(\epsilon_{\vec{k}+\vec{k}_1-\vec{k}'}))+\right.$$

$$\left.+(1-f(\epsilon_{k_1}))f(\epsilon_{k'})f(\epsilon_{\vec{k}+\vec{k}_1-\vec{k}'})\right]$$

$$d^3k = dS_k \, \frac{d\epsilon_k}{v_k}$$

$$d\epsilon_k = v\, dk$$



$$1/\tau_k \cong \frac{2\pi}{\hbar} V_0^2 \int dS_{k_1}\, dS_{k'}\, \frac{d\epsilon_{k_1}\, d\epsilon_{k'}}{v_{k_1}\, v_{k'}}$$

$$\delta(\epsilon_k + \epsilon_{k_1} - \epsilon_{k'} - \epsilon_{\vec{k}+\vec{k}_1-\vec{k}'}) \Big[f(\epsilon_{k_1})(1-f(\epsilon_{k'}))(1-f(\epsilon_{\vec{k}+\vec{k}_1-\vec{k}'}))$$

$$+ (1-f(\epsilon_{k_1})) f(\epsilon_{k'}) f(\epsilon_{\vec{k}+\vec{k}_1-\vec{k}'}) \Big] =$$

$$= \frac{2\pi V_0^2}{\hbar} \int \frac{d\epsilon_{k_1}\, d\epsilon_{k'}}{v_{k_1}\, v_{k'}} \Big[f(\epsilon_{k_1})(1-f(\epsilon_{k'}))(1-f(\epsilon_k + \epsilon_{k_1} - \epsilon_{k'}))$$

$$+ (1-f(\epsilon_{k_1})) f(\epsilon_{k'}) f(\epsilon_k + \epsilon_{k_1} - \epsilon_{k'}) \Big] \times$$

$$\times \int dS_{k_1}\, dS_{k'}\, \delta(\epsilon_k + \epsilon_{k_1} - \epsilon_{k'} - \epsilon_{\vec{k}+\vec{k}_1-\vec{k}'}) \simeq$$

$$\simeq \frac{2\pi V_0^2}{\hbar v_F^2} \int d\epsilon_1\, d\epsilon' \Big[f(\epsilon_1)(1-f(\epsilon'))(1-f(\epsilon+\epsilon_1-\epsilon')) +$$

$$+ (1-f(\epsilon_1)) f(\epsilon') f(\epsilon+\epsilon_1-\epsilon') \Big] \times$$

$$\times \int dS_{k_1}\, dS_{k'}\, \delta(\epsilon+\epsilon_1-\epsilon'-\epsilon_{\vec{k}+\vec{k}_1-\vec{k}'})$$

where $\epsilon \equiv \epsilon_k$

Consider the limit $\underline{T=0\ ,\ \epsilon>0}$

First term:

$$f(\epsilon_1)(1-f(\epsilon'))(1-f_{\epsilon+\epsilon_1-\epsilon'}) \to \begin{cases}\epsilon_1<0\\ \epsilon'>0\\ \epsilon+\epsilon_1-\epsilon'>0\end{cases}$$

$$\epsilon' > -\epsilon-\epsilon_1$$

$$0>\epsilon_1>\epsilon'-\epsilon$$

$$\epsilon>\epsilon'$$

First term contributes

$$\int_0^{\epsilon} d\epsilon' \int_{\epsilon'-\epsilon}^{0} d\epsilon_1 \int dS_{k_1}\, dS_{k'}\, \delta(\epsilon+\epsilon_1-\epsilon'-\epsilon_{k+k_1-k'})$$

$$=\int_0^{\epsilon} d\epsilon' \int_{\epsilon'-\epsilon}^{0} d\epsilon_1 \int dS_{k_1}\, dS_{k'}\, \delta\left(\epsilon+\epsilon_1-\epsilon'-\frac{(\vec{k}+\vec{k}_1-\vec{k}')^2}{2m}\right)$$

$$=\int_0^{\epsilon} d\epsilon' \int_{\epsilon'-\epsilon}^{0} d\epsilon_1 \int dS_{k_1}\, dS_{k'}\, \delta\left(-2\epsilon'-\frac{\vec{k}\cdot\vec{k}_1-\vec{k}\cdot\vec{k}'-\vec{k}_1\vec{k}'}{2m}\cdot 2\right.$$

$$=\int_0^{\epsilon} d\epsilon' \int_{\epsilon'-\epsilon}^{0} d\epsilon_1 \int dS_{k_1}\, dS_{k'}\, \frac{2m}{2}\, \delta\left(2m\epsilon+\vec{k}\vec{k}_1-\vec{k}\vec{k}'-\vec{k}\right.$$

$$\simeq m\int_0^{\epsilon} d\epsilon' \int_{\epsilon'-\epsilon}^{0} d\epsilon_1 \int dS_{k_1}\, dS_{k'}\, \delta(\vec{k}\cdot\vec{k}_1-\vec{k}\vec{k}'-\vec{k}_1\vec{k}')$$

$$\int_0^{\epsilon} d\epsilon' \int_{\epsilon'-\epsilon}^{0} d\epsilon_1 = \int_0^{\epsilon} d\epsilon'\,(\epsilon-\epsilon') = \left(\epsilon\epsilon'-\frac{\epsilon'^2}{2}\right)\Big|_0^{\epsilon} =$$

$$=\frac{\epsilon^2}{2}$$

The contribution of the first term into

$$1/\tau_k = \frac{2\pi}{\hbar} U_0^2 \frac{\epsilon^2}{2} \times I$$

where $I = \int dS_{k_1}\, dS_{k'}\, \delta(\vec{k}\cdot\vec{k}_1 - \vec{k}\cdot\vec{k}' - \vec{k}_1\cdot\vec{k})$

We will show below that I is k independent (for $\vec{k}$ close to k_F) and therefore

$$1/\tau(\epsilon) \simeq \epsilon^2 U_0$$

This is a central point of Landau Fermi-liquid theory.

If one compares ϵ with $1/\tau(\epsilon)$

$\epsilon \overset{!}{>} 1/\tau(\epsilon)$ well resolved quasiparticles

$\epsilon > \epsilon^2 U_0$

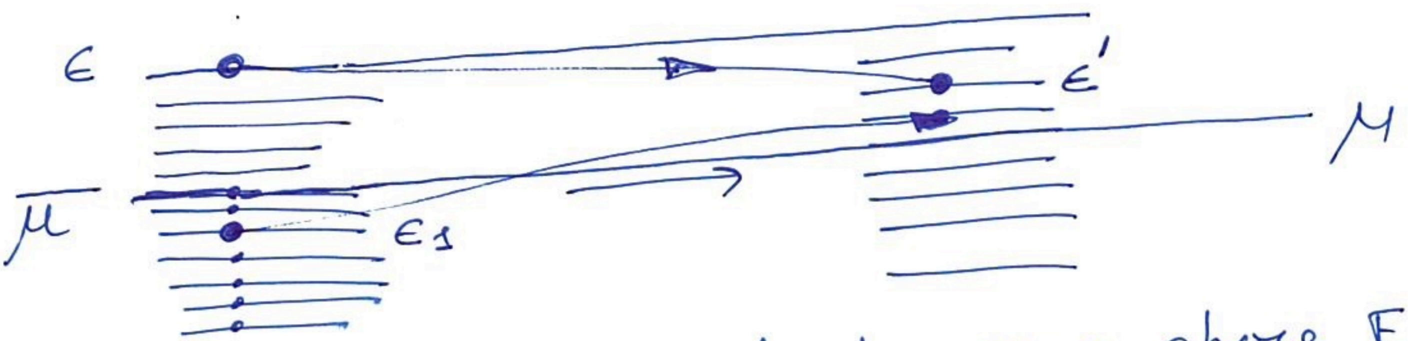
$1 > \epsilon U_0$ satified for $\epsilon \ll 1/U_0$, i.e. close to the Fermi surface.

The main conclusion: Pauli blocking makes electrons to be good quasi-particles near F.S

This was derived for the weak interaction.

The main argument boils down to the available for scattering phase space.

For the process at $T=0$, $\epsilon>0$



$-\epsilon < \epsilon_1 < 0$ in order to come above F.S.

$0 < \epsilon' < \epsilon$ At the end electron is aboe F.S. and below ϵ.

Thererefore ϵ^2 suppression of the scattering rate.

This holds due to the mere existance of the F.S. Therefore one can consistenly use quasi-particle picture near the F.S. for strong interaction, where perturbation theory does not hold. This is called Landau-Fermi liquid theory.

Note, that very strong interaction destroyes this picture and Fermi surface is completly removed.

This may happen, for example if Coulomb interaction is dominant and electrons ~~become cry.~~ form a crystall in real space - so called Wigner crystal transition.

There are many other ~~p.~~ transition. Typically involving "spontaneous" breaking of translational symmetry and driving system into a "non-Fermi liquid" states.

Next we check that the approximation of prefactor I by a constant was correct.

$$\int dS_{k_1}\, dS_{k'}\, \delta\left(\vec{k}\cdot\vec{k}_1 - \vec{k}\,\vec{k}' - \vec{k}_1\cdot\vec{k}'\right) =$$

$$= \int dS_{k'}\, dS_{k_1}\, \delta\left(\vec{k}_1\,(\vec{k}-\vec{k}') - \vec{k}\cdot\vec{k}'\right) =$$

$$\cancel{= \int k_1^2 \sin\theta_1\, d\theta_1\, d\varphi_1\, k'^2 \sin\theta'\, d\theta'\, d\varphi'}$$

$$\cancel{\delta(k}$$

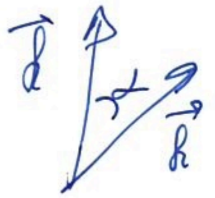
$$= \int dS_{k'} \int k_1^2 \int_0^\pi \sin\theta_1\, d\theta_1\, d\varphi_1\, \delta\left(k_1 \cos\theta\, |\vec{k}-\vec{k}'| - \vec{k}\vec{k}'\right)$$

$$= \int dS_{k'}\, k_F^2 \int_{-1}^{1} dx\, 2\pi\, \delta\left(k_F\, x\, |\vec{k}-\vec{k}'| - \vec{k}\cdot\vec{k}'\right) = I$$

$$\cancel{= 2\pi k_F^2 \int dS_{k'} \frac{1}{k_F |\vec{k}-\vec{k}'|}}$$

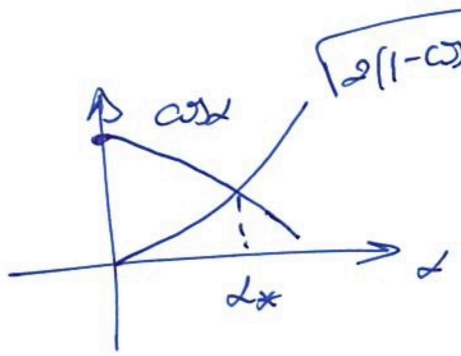
$$x\, k_F\, |\vec{k}-\vec{k}'| = \vec{k}\cdot\vec{k}'$$

$$x = \left| \frac{\vec{k}\cdot\vec{k}'}{|\vec{k}-\vec{k}'|\, k_F} \right| < 1$$



$$\left|\frac{\vec{k}\cdot\vec{k}'}{|\vec{k}-\vec{k}'|\,k_F}\right| = \left|\frac{\cos\alpha}{\sqrt{2-2\cos\alpha}}\right| < 1$$

$$|\cos\alpha| < \sqrt{2(1-\cos\alpha)}$$



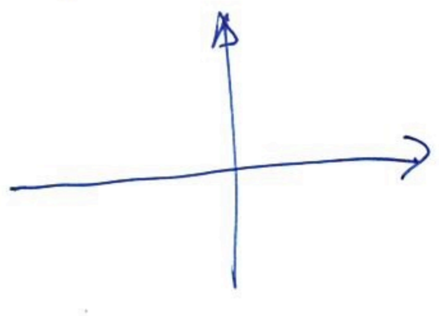
$p = \cos\alpha$

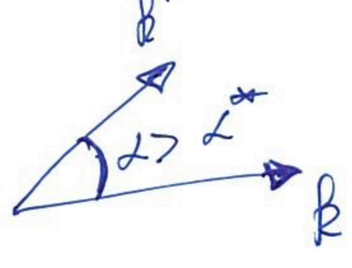
$p^2 < 2(1-p)$

$p^2 + 2p - 2 < 0$

$$p = \left[-1 \pm \sqrt{1+2}\right] = -1 \pm \sqrt{3} = \cos\alpha$$

$$\alpha > \alpha_* = \mathrm{Arccos}(-1+\sqrt{3}) \simeq 0.$$





$$I = \int_{\alpha > \alpha^*} dS_{k'}\ k_F^2\ 2\pi\ \frac{1}{k_F |\vec{k} - \vec{k}'|} =$$

$$= \frac{2\pi k_F^2}{k_F} \int_{\alpha > \alpha^*} dS_{k'}\ \frac{1}{k_F \sqrt{2 - 2\cos\alpha}} =$$

$$= 2\pi \int_{\alpha > \alpha^*} \sin\alpha\, d\alpha \int_0^{2\pi} d\varphi'\ \frac{1}{\sqrt{2(1-\cos\alpha)}} =$$

$$= (2\pi)^2 \int_0^{\sqrt{3}-1} \frac{dx}{\sqrt{2(1-x)}} = (2\pi)^2 \left(\sqrt{2} - \sqrt{4 - 2\sqrt{3}}\right) =$$

$$\cong (2\pi)^2\ 0.68$$

Therefore $I \simeq (2\pi)^2 \cdot 0.68$, i.e. constant, as claimed above.

Next we consider a limit of $T \gg \epsilon$ still small in comparison with Fermi energy
($\epsilon_F \gg T \gg \epsilon$).

In this case the window of integration is determined by T and

$$1/\tau_R \propto V_0^2 T^2$$

This results hold for d=3 and "practically" correct for d=2.
The phase space counting argument does not hold for d=1, quantum wires, edge states,...
The situation at <u>d=1</u> is more subtle.
It is used to be an example of the breaking of Landau-Fermi liquid theory. However, recent understanding, that it is practically LFT, but formulated in terms of different "composite" quasi-particles, that are non-trivially related to the original electrons:
"Fermi-Luttinger liquid", M. Khodas et. all PRB 2007.