Collisions and conservation laws: from kin. eq. to the classical hydrodynamics.

We have discussed the following coll.

1) electron + static disorder
2.) electron - electron
3) electron - phonon.

What are the conserved quantities?

For electron scattering of random potential there are two conserved quantities: the energy of electron at given energy and the number of electrons at given energy.

For the e-e scattering there are 3 conserved modes: the total energy, the total momentum and the number of electrons.

For e-ph scattering is only the number of electrons that is conserved.

More formaly: disorder scattering

$$\int_{\epsilon_p=\epsilon} I_{dis}[f(p,r,t)]\, d^d p = 0$$ ← on energy shell

$$\int_{\epsilon_p=\epsilon} I_{dis}[f(p,r,t)]\, d^d p\, \epsilon_p = 0$$

electron-electron scattering

$$\int I_{ee}[f]\, d^d p = 0$$

$$\int I_{ee}[f]\, p\, d^d p = 0$$

$$\int I_{ee}[f]\, \epsilon\, d^3 p = 0$$

electron-phonon scattering

$$\int I_{eph}[f]\, d^d p = 0$$

Note that the corresponding modes

$\rho(r,t) = \int d^d p\, f(p,r,t)$ density of particles

$\underline{P}(r,t) = \int d^d p\, p\, f(p,r,t)$ density of ~~total~~ momentum of electrons

$E(p,r,t) = \int d^3 p\, \epsilon_p f(p,r,t)$ density of total energy

are hydrodynamic modes of the classical fluid.

To get the set of hydrodynamic eqs we start with Boltzmann equation

$$\frac{\partial f}{\partial t} + \vec{v}_p \nabla_r f + \vec{F} \cdot \nabla_p f = I_{ee}[f]$$

let us ignore this term for "simlicity", i.e. assume that there are no external field.

$$\frac{\partial}{\partial t} \underbrace{\int d^3p\, f(p)\, m}_{\rho} + \nabla_r \underbrace{\int d^3p\, m v_p \cdot f}_{j} = 0$$

density flux or (mass flux if multiplied by m)

$$\frac{\partial}{\partial t}\rho + \operatorname{div} \vec{j} = 0$$

or by defining V as $\vec{j} = \rho \vec{V}$

we get the continuity eq.

$$\boxed{\frac{\partial}{\partial t}\rho + \operatorname{div}(\rho \vec{V}) = 0}$$

To get the Navier-Stockes eq. one multiplies Boltzmann eq. by $m\vec{v}_\alpha$

$$\frac{\partial}{\partial t}\underbrace{\int (d^3p)\, m v_\alpha(p)\, f(p)}_{j_\alpha = \rho V_\alpha} + \nabla_r^\beta \underbrace{\int \Big(v_\alpha(p) v_\beta(p) f(p) m \Big) (d^3 p)}_{\Pi_{\alpha\beta}} = 0$$

momentum flux

$$\boxed{\frac{\partial}{\partial t}(\rho V_\alpha) + \nabla_r^\beta \, \Pi_{\alpha\beta}(r,t) = 0}$$

Multiplying by ϵ_p and integrating we get

$$\frac{\partial}{\partial t}\underbrace{\int d^dp\, \epsilon_p f(p,r,t)}_{\substack{E(r,t)\\ \text{energy density}}} + \nabla_r^i \underbrace{\int d^3p\, \epsilon_p v_p f(p,r,t)}_{\vec{q}(r,t) \text{ flux of energy}} = 0$$

$$\boxed{\frac{\partial}{\partial t} E(r,t) + \operatorname{div} \vec{q}(r,t) = 0}$$

These 3 equations should hold regardless of how accurately we have computed the collision integral. However these eqs are not explicit, since we do not know how to relate $E, \Pi_{\alpha,\beta}, \vec{q}$ to the observable macroscopic quantities.

To establish such a relation consider electrons at temperature T at spatial scales much larger than $l_{ee}(T)$.

This means that $I_{ee}[f]$ is large, unless distribution function is choosen properly.

In other word if $I_{ee}[f] \neq 0$ one may estimate $I_{ee}[f] = \frac{f(p) - f_{FD}(p)}{\tau_{ee}(T)} > \frac{\partial}{\partial t} f \simeq \omega f$

So for $\tau_{ee}\,\omega \ll 1$ or $t \gg \tau_{ee}$ the $I_{ee}[f]$ is the biggest term in the B. E. Unless we choose distribution function along the direction where it vanishes.

Note, $f_{FD}(p) = \frac{1}{e^{\frac{\epsilon_p - \mu}{T}} + 1}$

The function

$$f(p) = \frac{1}{e^{\frac{\epsilon_p - V(x)p - \mu(x)}{T(x)}} + 1}$$

corresponds to a boost with velocity $V(x)$, change of local temperature and density

If the collision integral is local (on the scale $\mu(x)$, $T(x)$, $V(x)$ varies) this function nullifies the coll. integral.

Let us compute the hydrodynamic variables.

$$\rho(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\frac{\epsilon_p - Vp - \mu(x)}{T(x)}} + 1} =$$

$$\cong \int^{p_F} \frac{d^3p}{(2\pi)^3} + \left(T/\epsilon_F\right) = \frac{1}{(2\pi)^3} \frac{4\pi p_F^3}{3} + \left(T/\epsilon_F\right)$$

$$p_F = \sqrt{2m\mu}$$

$$j_\alpha = \int \frac{d^3p}{(2\pi)^3} v_\alpha f(p) = \frac{1}{m} \int \frac{d^3p}{(2\pi)^3} \frac{p_\alpha}{e^{\frac{\epsilon_p - Vp - \mu}{T}} + 1} =$$

$$\frac{p^2}{2m} - Vp = \frac{(p - mV)^2}{2m} - \frac{mV^2}{2}$$

$$= \frac{1}{m} \int \frac{d^3p}{(2\pi)^3} \left(\vec{p}_\alpha + mV_\alpha\right) f\left(\epsilon_p - \mu - \frac{mV^2}{2}\right) =$$

Shifting p

$$= V_\alpha \rho = V_\alpha(x,t)\, \rho(x,t)$$

Assume $V^2/m \ll \mu$ for simplicity

$$\Pi_{\alpha\beta} = m\int (d^3p)\, v_\alpha v_\beta\, f_{Fb}\left(\frac{\epsilon_p - V\cdot p - \mu}{T}\right) =$$

$$= \frac{m}{m^2}\int (d^3p)\, p_\alpha p_\beta\, f\left(\frac{\epsilon_p - Vp - \mu}{T}\right) =$$

$$= \frac{m}{m^2}\int (d^3p)(p_\alpha + mV_\alpha)(p_\beta + mV_\beta)\, f\left(\frac{\epsilon_p - \mu}{T}\right) =$$

$$= \frac{m}{m^2}\int (d^3p)\Big(p_\alpha p_\beta \underbrace{+ mV_\alpha p_\beta + mV_\beta p_\alpha}_{=0} + m^2 V_\alpha V_\beta\Big) f\left(\frac{\epsilon_p - \mu}{T}\right) =$$

$$= \underbrace{\frac{1}{m}\int (d^3p)\, p_\alpha p_\beta\, f\left(\frac{\epsilon_\mu - \mu}{T}\right)}_{P(\rho)} + V_\alpha V_\beta \cdot \rho$$

ρ — mass density

$P(\rho)$ ← pressure

$$P(\rho) = \frac{\delta_{\alpha,\beta}}{m^2}\frac{1}{(2\pi)^3}(2\pi)\cdot 2\int_0^{p_F} p^4\,dp = \frac{\delta_{\alpha\beta}}{m^2 2\pi^2}\frac{p_F^5}{5} =$$

$$= \frac{\delta_{\alpha\beta}}{10\pi^2 m^2}\, p_F^5 = \frac{\delta_{\alpha\beta}}{10\pi^2 m^2}(6\pi\rho)^{5/3}$$

$$\rho = \frac{p_F^3\, m}{6\pi^2} \rightarrow p_F = \left(\frac{6\pi^2}{m}\rho\right)^{1/3}$$

This are equation that describe the hydrod. of weakly interacting 3D degenerate electrons.

By considering $T \gg \epsilon_F$ we get the hydrod. of

classical electronic fluid.
However this equation lacks friction.
This means that we need to correct the eqs.

Consider

$$f(p,r,t) = f_{FD}\left(\frac{\epsilon_p - Vp - \mu(\vec{z})}{T(z)}\right) + \delta f$$

$$\frac{\partial}{\partial t} f(p,r,t) + v\frac{\partial}{\partial r} f = I_{ee}[\delta f] \simeq \frac{\delta f}{\tau_{ee}}$$

$$z \equiv \frac{\epsilon_p - \vec{V}\vec{p} - \mu(z)}{T(z)}$$

$$\text{l.h.s.} \simeq \frac{\partial}{\partial z} f_{FD}(z)\left(\frac{\partial}{\partial t} z + v_p \frac{\partial}{\partial \vec{z}} z\right) \simeq \frac{\delta f}{\tau_{ee}}$$

$$\delta f \simeq \tau_{ee}\left(\frac{\partial z}{\partial t} + v_p \frac{\partial z}{\partial \vec{z}}\right)\frac{\partial}{\partial z} f_{FD}(z)$$

Correction to the momentum flux - "viscous part of the stress tensor"

in LL notations

$$-\sigma'_{\alpha\beta} \equiv \delta\Pi_{\alpha\beta} = m\int(d^3p)\, v_\alpha v_\beta f(p) = \int(d^3p)\, \frac{p_\alpha p_\beta}{m^2}\tau_{ee}(p)$$

$$\left(\frac{\partial z}{\partial z} + \vec{v}_p \frac{\partial z}{\partial \vec{z}}\right)\frac{\partial}{\partial z} f_{FD} \simeq$$

$$\simeq \frac{m\tau_{ee}}{m^2}\int(d^3p)\, \vec{p}_\alpha \vec{p}_\beta \frac{\vec{p}}{m}\frac{\partial}{\partial \vec{r}}\frac{\epsilon_p - \vec{V}\cdot\vec{p} - \mu}{T}\frac{\partial}{\partial z} f =$$

$$= \frac{m\tau_{ee}}{m^3}\int(d^3p)\, p_\alpha p_\beta p_\gamma \frac{\partial V_\delta}{\partial r_\gamma} p_\delta \frac{-1}{T}\frac{\partial}{\partial z} f(z)$$

$$= \frac{m T_L ee}{m^3} \int (d^3p)\, p_\alpha p_\beta p_\gamma p_\delta \frac{\partial V_\delta}{\partial r_\gamma} \frac{\partial}{\partial \epsilon} f_{FD}(\epsilon) =$$

$$\frac{\partial \xi}{\partial z} = \frac{\partial \xi}{\partial \epsilon} \frac{\partial \epsilon}{\partial z} = T \frac{\partial}{\partial \epsilon}$$

$$\cancel{= \frac{1}{m^3} \int d\epsilon}$$

$$= \frac{T_{ee}}{m^2} \frac{\partial V_\delta}{\partial r_\gamma} \int (d^3p)\, p_\alpha p_\beta p_\gamma p_\delta \frac{\partial f}{\partial \epsilon} =$$

$$= \frac{T_{ee}}{m^2} \frac{\partial V_\delta}{\partial r_\gamma} \delta_{\alpha\beta} \int (d^3p)\, p_\alpha^2 p_\gamma p_\delta \frac{\partial f}{\partial \epsilon} +$$

$$+ \frac{T_{ee}}{m^2} \frac{\partial V_\delta}{\partial r_\gamma} \delta_{\alpha\neq\beta} \int (d^3p) \left[\delta_{\alpha\gamma} \delta_{\beta\delta} p_\alpha^2 p_\beta^2 + \delta_{\alpha\delta} \delta_{\beta\gamma} p_\alpha^2 p_\beta^2 \right] \frac{\partial f}{\partial \epsilon} =$$

$$= \frac{T_{ee}}{m^2} \frac{\partial V_\delta}{\partial r_\gamma} \Big\{ \delta_{\alpha\beta} \delta_{\gamma\delta} \int (d^3p) \left(p_\alpha^2 p_\delta^2 \frac{\partial f}{\partial \epsilon} \right) +$$

$$\delta_{\alpha\neq\beta} \int d^3p \left[\delta_{\alpha\gamma} \delta_{\beta\delta} \cancel{p_\alpha^2 p_\beta^2} + \delta_{\alpha\delta} \delta_{\beta\gamma} \right] p_\alpha^2 p_\beta^2 \frac{\partial f}{\partial \epsilon} =$$

$$= \eta \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} - \frac{2}{3} \delta_{\alpha\beta} \frac{\partial v_\alpha}{\partial x_\alpha} \right) + \zeta \delta_{\alpha\beta} \frac{\partial v_\alpha}{\partial x_\alpha}$$

$$\eta \sim \frac{\tau_{ee}}{m^{\cancel{2}}} \int (d^3p)\, p^4 \frac{\partial f}{\partial \epsilon} =$$

$$= \frac{\tau_{ee}}{m^{\cancel{2}}} \int d\epsilon\, \nu(\epsilon) \frac{\partial f}{\partial \epsilon} p^4(\epsilon) =$$

for $T \ll \epsilon_F$

$$\simeq \frac{\tau_{ee}\, p_F^4 \nu}{m^{\cancel{2}}} = \tau_{ee} \frac{p_F}{m} \frac{p_F^3 \nu}{m^{\cancel{2}}} = \tau_{ee}\, v_F\, m$$

$$\frac{v_F^3 m^3 \cancel{m^2} v_F}{\cancel{m^2}} =$$

$$\nu(\epsilon) = \int dp\, p^2 \delta\left(\epsilon - \frac{p^2}{2m}\right) = \frac{p_F^2}{p/m} = p_F m$$

$$= l_{ee} v_F^4 m^4 = l_{ee} v_F m p_F^3 = l_{ee} v_F \rho$$

ρ mass density

$$l_{ee} \sim v_F \tau_{ee} \sim v_F \frac{\epsilon_F}{T^2} \sim \frac{m v_F^3}{T^2}$$

$$\tau_{ee}^{-1} \sim T^2/\epsilon_F \qquad \epsilon_F \sim m v_F^2$$

$$\eta \sim \frac{m^2 v_F^4 \rho}{T^2} \xrightarrow[T \to 0]{} \infty$$

checking units

Units

$$[\sigma'] = \rho v^2 = p = F/m^2 = \frac{kg\,m}{sec^2 \cdot m^2} = $$

$$= \frac{kg}{m \cdot sec^2}$$

$$\frac{kg}{m^3}\left(\frac{m}{sec}\right)^2 = \frac{kg}{m\,sec^2}$$

$$\sigma' \sim \eta \frac{\partial v}{\partial x} = \eta \frac{m}{sec}\frac{1}{m} = \eta/sec = \frac{kg}{m\,sec^2}$$

$$\eta = \frac{kg}{m \cdot sec}$$

$$\eta = l v \rho = m \frac{m}{sec}\frac{kg}{m^3} = \frac{kg}{m\,sec}$$

mass density

OK

$$[\Pi] = p^3 \frac{m^2 kg}{sec^2} = \frac{p}{sec^2} = \frac{kg\,m^2}{sec^3} =$$

$$= \frac{kg}{m\,sec^2} \cdot \left(\frac{m^2 kg}{sec}\right) = \frac{kg}{m\,sec}$$

$$\hbar$$

$$[\hbar] = p x = \frac{kg\,m}{sec} m = \frac{kg\,m^2}{sec}$$