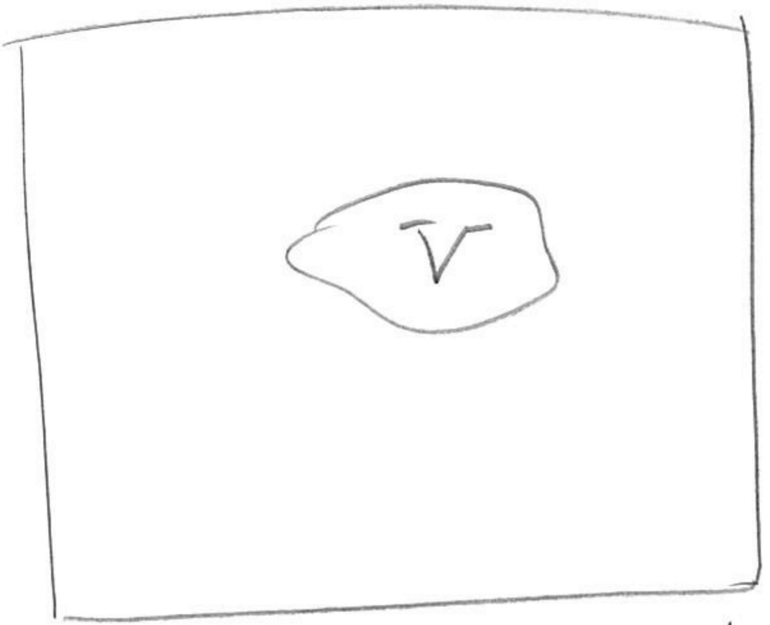
Poisson processes – a reminder



N_0

number of particles

V_0 total volume

$\frac{V}{V_0}$ a probability for a particle to be in V

binomial distribution

$$\omega_N = \frac{N_0!}{N!\,(N_0-N)!}\left(\frac{V}{V_0}\right)^{N}\left(1-\frac{V}{V_0}\right)^{N_0-N}$$

probability that N particles are in V

In the limit $\frac{V}{V_0} \to 0$, $N_0 \to \infty$

$$\frac{V}{V_0} N_0 \to \overline{N}$$

$$\omega_N \simeq \frac{\bar{N}^N}{N!}\left(1-\frac{\bar{N}}{N_0}\right)^{N_0} \simeq \frac{\bar{N}^N e^{-\bar{N}}}{N!}$$

poisson distribution

$$\sum_N \omega_N = 1$$

$$\langle N^2 \rangle = \sum_{N=0}^{\infty} N^2 \omega_N =$$

$$= e^{-\bar{N}} \sum_{N=1}^{\infty} \frac{\bar{N}^N N}{(N-1)!} = \bar{N}^2 + \bar{N}$$

$$\langle N^2 \rangle - \langle N \rangle^2 = \bar{N}$$

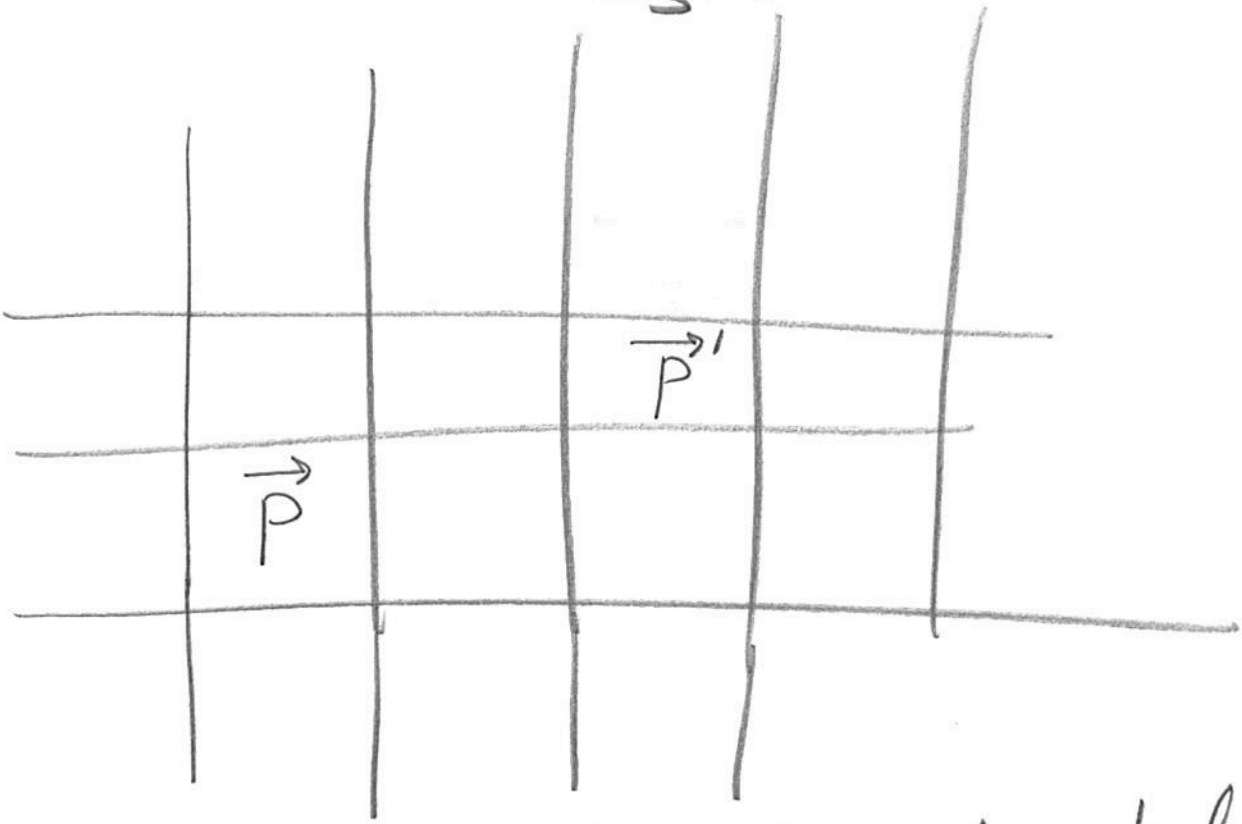
In fad all cummulants of $N = \langle N \rangle$

Boltzmann- Langevin Eq

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla_r + e\vec{E}\,\nabla_p - \hat{I}\right)\delta f(p,r,t) = \delta J(p,r,t)$$

Langevin flux

$$f(p,r,t) = \langle f(p,r,t) \rangle + \delta f(p,r,t)$$



$\delta J(p,r,t)$ random flux of particles arising from fluctuation into and out of the state (p,r)

Let us consider a flux of particles that flows from the state

$p,r \longrightarrow p',r$ (the collisions are assumed to be local in space $r=r'$)

$J(p,p';r,t)$

similarly there is an opposite flux

$p,r \longleftarrow p',r$

$J(p',p;r,t)$

The fluctuations of these fluxes we denote $\delta J(p, p', r, t)$ and $\delta J(p', p, r, t)$

The net flux into the state p

$$\delta J(p, r, t) = \sum_{p'} \left[\delta J(p', p, r, t) - \delta J(p, p', r, t) \right]$$

$$= \underbrace{\sum_{p} \delta J(p', p, r, t)}_{\delta J^{+}(p, r, t)} - \underbrace{\sum_{p} \delta J(p, p', r, t)}_{\delta J^{-}(p, r, t)}$$

To define the Boltzmann-Langevin eq we need to define the statistics of $\delta J(p, r, t)$. The second order corr. funct.

(*)

$$\langle \delta J(p_1, r_1, t_1)\, \delta J(p_2, r_2, t_2) \rangle =$$

$$\langle (\delta J^{+}(p_1, r_1, t_1) - \delta J^{-}(p_1, r_1, t_1))(\delta J^{+}(p_2, r_2, t_2) - \delta J^{-}(p_2, r_2, t_2)) \rangle$$

$$= \langle \delta J^{+}(p_1, r_1, t_1)\, \delta J^{+}(p_2, r_2, t_2) \rangle + \langle \delta J^{-}(p_1, r_1, t_1)\, \delta J^{-}(p_2, r_2, t_2) \rangle - \langle \delta J^{-}(p_1, r_1, t_1)\, \delta J^{+}(p_2, r_2, t_2) \rangle$$

$$- \langle \delta J^{+}(p_1, r_1, t_1)\, \delta J^{-}(p_2, r_2, t_2) \rangle$$

The random elementary processs is poissonian

$$\left\langle \delta J(\vec{p}_1', \vec{p}_1; \vec{r}_1, t_1)\, \delta J(\vec{p}_2', \vec{p}_2; \vec{r}_2, t_2) \right\rangle =$$

$$= \delta(\vec{p}_1' - \vec{p}_2')\, \delta(\vec{p}_1 - \vec{p}_2)\, \underbrace{\delta(\vec{r}_1 - \vec{r}_2)\, \delta(t_1 - t_2)}_{\text{locality in space and time}}\, \overline{J}(\vec{p}_1' \to \vec{p}_1)$$

$$\langle \delta J^+(p_1, r_1, t_1)\, \delta J^+(p_2, r_2, t_2) \rangle = \sum_{p_1' p_2'} \langle \delta J(p_1', p_1, r_1, t_1)\, \delta J(p_2', p_2, r_2, t_2) \rangle$$

$$= \sum_{p_1', p_2'} \delta_{p_1' - p_2'}\, \delta_{p_1 - p_2}\, \delta(r_1 - r_2)\, \delta(t_1 - t_2)\, \overline{J}(p_1' \to p_1) =$$

$$= \delta_{p_1 - p_2}\, \delta(\vec{r}_1 - \vec{r}_2)\, \delta(t_1 - t_2) \sum_{p_1'} \overline{J}(p_1' \to p_1)$$

$$\langle \delta J^-(p_1, r_1, t_1)\, \delta J^-(p_2, r_2, t_2) \rangle = \sum_{p_1', p_2'} \langle \delta J(p_1, p_1', r_1, t_1)\, \delta J(p_2, p_2', r_2, t_2) \rangle$$

$$= \sum_{p_1', p_2'} \delta(p_1 - p_2)\, \delta(p_1' - p_2')\, \delta(r_1 - r_2)\, \delta(t_1 - t_2)\, \overline{J}(p_1 \to p_1') =$$

$$= \delta(p_1 - p_2)\, \delta(r_1 - r_2)\, \delta(t_1 - t_2) \sum_{p_1'} \overline{J}(p_1 \to p_1')$$

$$\langle \delta J^{+}(p_1,r_1,t_1)\,\delta J^{-}(p_2,r_2,t_2)\rangle =$$

$$= \sum_{p_1' p_2'} \left\langle \delta J(p_1',p_1,r_1,t_1)\,\delta J(p_2,p_2',r_2,t_2) \right\rangle =$$

$$= \sum_{p_1' p_2'} \delta(p_1'-p_2)\,\delta(p_1-p_2')\,\delta(r_1-r_2)\,\delta(t_1-t_2)\ \overline{J}(p_1' \to p_1) =$$

$$= \delta(r_1-r_2)\,\delta(t_1-t_2)\ \overline{J}(p_2 \to p_1)$$

$$\langle \delta J^{-}(p_1,r_1,t_1)\,\delta J^{+}(p_2,r_2,t_2)\rangle =$$

$$= \sum_{p_1' p_2'} \langle \delta J(p_1,p_1',r_1,t_1)\,\delta J(p_2',p_2,r_2,t_2)\rangle =$$

$$= \sum_{p_1' p_2'} \delta(p_1-p_2')\,\delta(p_1'-p_2)\,\delta(r_1-r_2)\,\delta(t_1-t_2)\ \overline{J}(p_1 \to p_1') =$$

$$= \delta(r_1-r_2)\,\delta(t_1-t_2)\,\overline{J}(p_1 \to p_2)$$

Substitute into ⊛ we get

$$\langle \delta J(p_1,r_1,t_1)\,\delta J(p_2,r_2,t_2)\rangle = \delta(r_1-r_2)\,\delta(t_1-t_2)\Big\{$$

$$\delta(p_1-p_2)\sum_{p'}\left[\overline{J}(p' \to p_1) + \overline{J}(p_1 \to p')\right] -$$

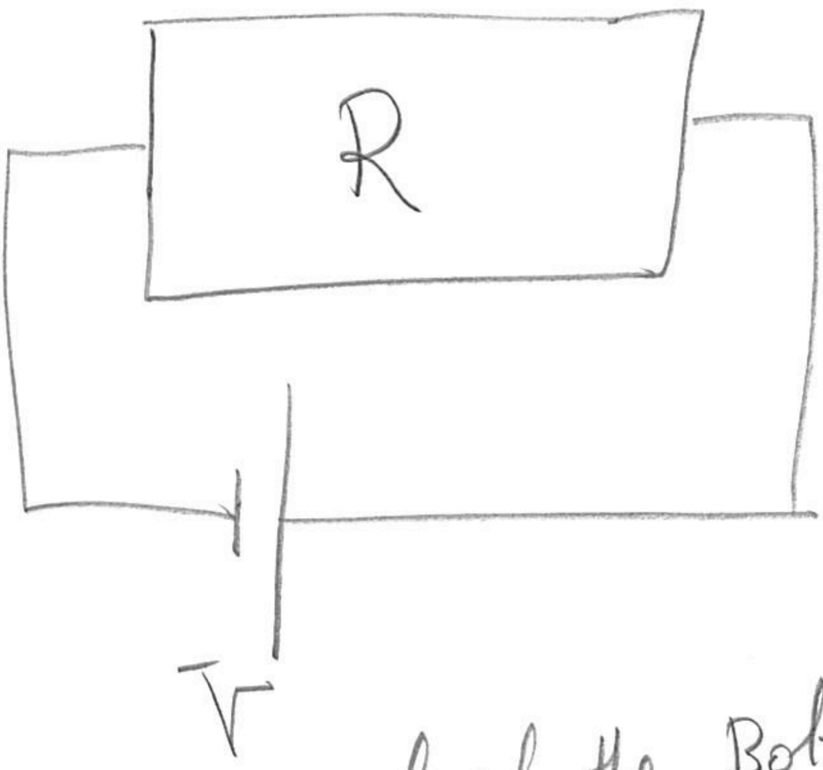
$$- \overline{J}(p_1 \to p_2) - \overline{J}(p_2 \to p_1)\Big\}$$

For the case of elastic scattering of electrons by static disorder

$$\bar{J}(p \to p') = W(p, p') f(p) (1 - f(p'))$$

average value of the distr. function $f(p, r, t)$

Now we compute the noise in the disordered metal



We have solved the Boltzmann eq. for this problem.

$$f(\epsilon, x) = \frac{x}{L} f_0\left(\epsilon - \frac{eV}{2}\right) + \left(1 - \frac{x}{L}\right) f_0\left(\epsilon + \frac{eV}{2}\right)$$

This satisfies the diff. equation

$D \partial_x^2 f(\epsilon, x) = 0$ with the boundary conditions

$$f(\epsilon, 0) = f_0\left(\epsilon + \frac{eV}{2}\right)$$

$$f(\epsilon, L) = f_0\left(\epsilon - \frac{eV}{2}\right)$$

and dc. current density

$$j = e\nu \int \nabla f \, (d\epsilon) = -\frac{eD\nu}{L} \int d\epsilon \left[f\left(\epsilon - \frac{eV}{2}\right) - f\left(\epsilon + \frac{eV}{2}\right)\right]$$

$$= \frac{D\nu}{L} e^2 V$$

Total curren

$$I = A \cdot j = e^2 \nu \frac{DA}{L} V = GV$$

$$G = e^2 \nu D \frac{A}{L}$$

cross-section

Current noise

$$\delta j_\alpha(r,t) = e\int (dp)\, \vec{v}_\alpha(p)\, \delta f(p,r,t)$$

$$\langle \delta j_\alpha(r,t)\, \delta j_\beta(r',t')\rangle = e^2 \int (dp)(dp')\, v_\alpha(p)\, v_\beta(p')$$

$$\langle \delta f(p,r,t)\, \delta f(p',r',t')\rangle$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\nabla + \hat{I}\right)\delta f = \delta J^{ext}(p,r,t)$$

$$\delta f(p,r,t) = \int dp'\,dr'\,dt'\, \mathcal{D}(p,r,t;p',r',t')\,\delta J^{ext}(p',r',t')$$

where $\mathcal{D}$ is the propagator

$$\left(\frac{\partial}{\partial t} + \vec{v}_p\vec{\nabla} + \hat{I}\right)\mathcal{D}(p,r,t;p',r',t') = \delta(p-p')\,\delta(r-r')\,\delta(t-t')$$

linear. coll. integral

$$\langle \delta f(p_1,r_1,t_1)\,\delta f(p_2,r_2,t_2)\rangle = \int dp_1'\,dr_1'\,dt_1'\,dp_2'\,dr_2'\,dt_2'$$

$$\mathcal{D}(p_1,r_1,t_1;p_1',r_1',t_1')\,\mathcal{D}(p_2,r_2,t_2;p_2',r_2',t_2')\cdot$$

$$\cdot\langle \delta J^{ext}(p_1',r_1',t_1')\,\delta J^{ext}(p_2',r_2',t_2')\rangle$$

Assume elastic scattering + diffusion approximation

$f(p,r,t) \cong f(\varepsilon_p, r, t)$

$J(p \to p') = \omega(\vec{n}, \vec{n}')\, f(\varepsilon_p, r, t)(1 - f(\varepsilon_p, r, t))\,\delta_{\varepsilon_p - \varepsilon_{p'}}$

$\langle \delta J(p_1, r_1, t_1)\, \delta J(p_2, r_2, t_2) \rangle = 2\delta(r_1 - r_2)\,\delta(t_1 - t_2)\,\delta(\varepsilon_{p_1} - \varepsilon_{p_2})$

$\left[-W(n_1, n_2) + \frac{1}{\tau}\delta(n_1, n_2) \right] f(\varepsilon_{p_1}, r_1, t_1)(1 - f(\varepsilon_{p_1}, r_1, t_1))$

$\frac{1}{\tau} = \int d\vec{n}\, W(n, n')$ assumin $1/\tau$ independent on n (isotropic spectrum)

$\langle \delta f(p_1, r_1, t_1)\, \delta f(p_2, r_2, t_2) \rangle = \delta(\varepsilon_{p_1} - \varepsilon_{p_2}) \int dn_1' dn_2' dr' dt'$

$\mathcal{D}(n_1, r_1, t_1; n_1', r_1', t_1')\, \mathcal{D}(n_2, r_2, t_2; n_2', r_2', t_2')\,(-2)$

$\left[W(n_1', n_2') - \frac{1}{\tau}\delta(n_1' - n_2') \right] f(\varepsilon_{p_1}, r'', t'')(1 - f(\varepsilon_{p_1}, r'', t''))$

The diffuson propagator $\mathcal{D}$ satisfies the equation

$$\int dn_1' \left\{ \left(\frac{\partial}{\partial t_1} + \vec{v}_1 \nabla \right) \delta(\vec{n}_1 - \vec{n}_1') + \left[-W(\vec{n}_1, \vec{n}_1') + \frac{\delta(n_1 - n_1')}{\tau} \right] \right\}$$

$$\mathcal{D}(n_1, r_1, t_1; n_2, r_2, t_2) = \delta(n_1 - n_2) \delta(r_1 - r_2) \delta(t_1 - t_2)$$

$$\int dn_2' \left\{ - \left(\frac{\partial}{\partial t_1} + \vec{v}_1 \nabla \right) \delta(\vec{n}_2 - \vec{n}_2') + \left[-W(n_2, n_2') + \frac{\delta(n_2 - n_2')}{\tau} \right] \right\}$$

$$\mathcal{D}(n_1, r_1, t_1; n_2, r_2, t_2) = \delta(n_1 - n_2) \delta(r_1 - r_2) \delta(t_1 - t_2)$$

schematically

$$\mathcal{L}_1 \mathcal{D}(1,2) = \delta(1,2)$$

$$\mathcal{L}_2^{T} \mathcal{D}(1,2) = \delta(1,2)$$

Adding to the integral on the page 10 terms $\frac{\partial}{\partial t_1'} + v_1' \nabla_{r_1'}$ we complete the operator $w(n_1', n_2) - \frac{\delta(n_1' - n_2')}{\tau} \rightarrow \mathcal{L}^T$ since these are full derivative term acting on $\mathcal{D}$ under the integral with b.c. zero for $\mathcal{D}$ it does not change the integral.

$$\langle \delta f(1)\, \delta f(2) \rangle = \delta(\epsilon_{p_1} - \epsilon_{p_2}) \int dr' dt' \Big\{$$

$$\int dn_1'\, \mathcal{D}(n_1, r_1, t_1, n_1', r_1', t_1')\, \mathcal{L}^T \mathcal{D}(n_2, r_2, t_2, n_1', r_1', t_1') +$$

acts on second argument

$$+ \int dn_2' \left(\mathcal{L}^T \mathcal{D}(n_1, r_1, t_1; n_2', r_2', t_2') \right) \mathcal{D}(n_2, r_2, t_2; n_2', r_2', t_2') \Big\}$$

acts on this

$$f(1-f)_{\epsilon_p, r', t'} = \delta(\epsilon_{p_1} - \epsilon_{p_2}) \big[\mathcal{D}(1,2)\, f(2)(1-f(2))$$

$$+ f(1)(1-f(1))\, \mathcal{D}(2,1) \big]$$

The current fluctuations

$$\delta J_\alpha(r,t) = e \int (dp)\, v_\alpha\, \delta f(p, r, t) =$$

$$= e v_F \int dn\, d\epsilon_p\, n_\alpha\, \delta f(n, \epsilon_p, r, t)\, \nu(\epsilon_p)$$

$$\langle \delta J_\alpha(r_1,t_1)\, \delta J_\beta(r_2,t_2) \rangle = e^2 v_F^2 \int d\epsilon_p\, \nu(\epsilon_p)$$

$$\big[f(2)(1-f(2)) \langle n_\alpha \mathcal{D}\, n_\beta \rangle (1,2) +$$

$$f(1)(1-f(1)) \langle n_\alpha \mathcal{D}\, n_\beta \rangle (2,1) \big]$$

In the diffusive approximation

$$\langle n_\alpha \mathcal{D}\, n_\beta \rangle = \frac{\tau}{d}\,\delta(r_1-r_2)\,\delta(t_1-t_2)\,\delta_{\alpha\beta}$$

$$\langle \delta j_\alpha(1)\,\delta j_\beta(2)\rangle = \delta_{\alpha\beta}\,\delta(r_1-r_2)\,\delta(t_1-t_2)\,2\sigma$$

$$\int d\epsilon\, f(\epsilon,r_1)\,(1-f(\epsilon,r_1))$$

At equilibrium

$$f(\epsilon,r) = \frac{1}{e^{\frac{\epsilon-\mu}{T}}+1}$$

$$\int d\epsilon\, f(\epsilon)(1-f(\epsilon)) = T$$

Integrate the current density over the volume $\quad \delta I(t) \equiv \int d^3r\; j(r,t)\,\frac{1}{L}$

$$L^2\langle \delta I(t_1)\,\delta I(t_2)\rangle = \delta(t_1-t_2)\,2LA\sigma T$$

$$\langle \delta I(t_1)\,\delta I(t_2)\rangle = 2GT\,\delta(t_1-t_2)$$

Fourier

$$\langle \delta I\,\delta I\rangle_\omega = 2GT$$

agrees with FDT in a classical limit

$$= G\omega\,\mathrm{cth}\frac{\omega}{2T} \xrightarrow[\omega\to 0]{} 2TG$$

In the non-equilibrium situation ($T=0$)

$$f(\epsilon, x) = \frac{x}{L} f\left(\epsilon - \frac{eV}{2}\right) + \frac{L-x}{L} f\left(\epsilon + \frac{eV}{2}\right)$$

$$\int d\epsilon\, f(\epsilon, x)\left(1 - f(\epsilon, x)\right) = eV \frac{x(L-x)}{L^2}$$

$$\int dx \int d\epsilon\, f(\epsilon, x)\left(1 - f(\epsilon, x)\right) = \frac{eVL}{6}$$

$$\langle \delta I(t_1)\, \delta I(t_2) \rangle = \frac{1}{3} eGV\, \delta(t_1 - t_2)$$

Fourier $\rightarrow$ $\langle \delta I\, \delta I \rangle_\omega = \frac{1}{3} eGV = \frac{1}{3} eI$

<u>shot noise</u>

In general

$$\langle \delta I\, \delta I \rangle_\omega \equiv S_2 = \frac{2G}{L} \int d\epsilon\, dx\, f(\epsilon, x)\left(1 - f(\epsilon, x)\right)$$