Anomalous Hall effect

$$\partial_t f + \dot{\vec{r}}\cdot\nabla_r f + \dot{\vec{p}}\,\nabla_p f = I[f]$$

Berry curvature

$$\dot{r}_i = \frac{\partial \epsilon(p)}{\partial p_i} - \epsilon_{ijk}\dot{p}_j\,\Omega_{\zeta k} + \sum_{l'} \underbrace{W_{l'l}\,(\delta r_{l'l})_i}_{\text{side jumps accumulation}}$$

$$\dot{p}_i = e\left(E_i + \epsilon_{ijk}\dot{r}_j B_k/c\right)$$

$l = (p, \zeta)$

$l' = (p', \zeta')$

$$B_i = \epsilon_{ijk}\partial_{x_j} A_k$$ magnetic field

$$\Omega_{\zeta i} = \epsilon_{ijk}\partial_{p_j} A_{\zeta,k} = \frac{i}{2}\epsilon_{ijk}\left(\left\langle \frac{\partial u}{\partial k_j}\middle|\frac{\partial u}{\partial k_k}\right\rangle - \left\langle \frac{\partial u}{\partial k_k}\middle|\frac{\partial u}{\partial k_j}\right\rangle\right)$$

Berry curvature

$$I[f] = {\sum_{l'}}' \left[W_{ll'}f_{l'} - W_{l'l}f_l\right]$$

no Pauli blocking!

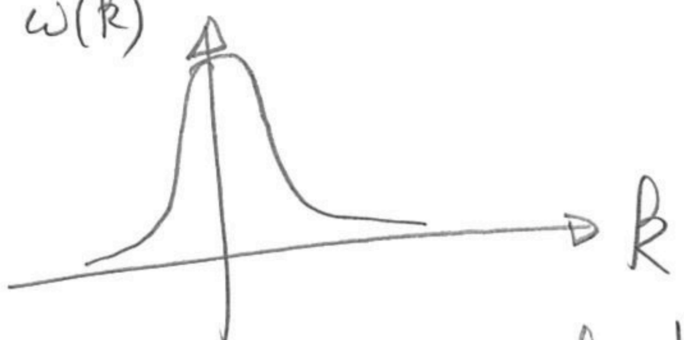
Semiclassical picture

Wave functions of H_0 (no disorder electric fields...)

$\Psi_k(r,t)$

The wave packet

$$\Psi_{k_0}(r,t) = \int dk\ \omega(k-k_0)\,\Psi_k(r,t)$$



The coordinate of the electron, i.e. the center of the wave function

$$r_c(k_0,t) = \int dr\ \Psi^*_{k_0}(r,t)\ r\ \Psi_{k_0}(r,t) =$$

$$= \int dr \int dk_1 dk_2\ \omega^*(k_1-k_0)\,\Psi^*_{k_1}(r,t)\ \mathbf{r}\ \omega(k_2-k_0)\,\Psi_{k_2}(r,t)$$

$$\Psi_k(r,t) = e^{-i\varepsilon_k t + ikr} u_k(r)$$

$$r_c = \int dr \int dk_1 dk_2 \; \omega^*(k_1-k_0)\,\omega(k_2-k_0)\, u^*_{k_1} u^*_{k_2}$$

$$e^{-i\varepsilon_{k_1} t + ik_1 r + i\varepsilon_{k_2} t - ik_2 r} \; r =$$

$$= \int dr \int dk_1 dk_2 \; \omega^*(k_1-k_0)\,\omega(k_2-k_0)\, e^{i(\varepsilon_{k_2}-\varepsilon_{k_1})t}$$

$$(-i)\left(\frac{\partial}{\partial k_1} e^{i(k_1-k_2)r}\right) u^*_{k_1}(r)\, u_{k_2}(r) =$$

$$\cancel{= \int dk_1 dk_2 (-i)\, \omega^*(k_1-k_0)\,\omega(k_2-k_0)\, e^{i(\varepsilon_{k_2}-\varepsilon_{k_1})t}}$$

$$\cancel{\left(\frac{\partial}{\partial k_1}\delta(k_1-k_2)\right)}$$

$$= i\int dr \int dk_1 dk_2 \; \omega^*_{k_1-k_0}\, \omega_{k_2-k_0}\, \cancel{e^{i(\varepsilon_{k_2}-\varepsilon_{k_1})t}}$$

$$\left(e^{i(k_1-k_2)r}\right) \frac{\partial}{\partial k_1} e^{i(\varepsilon_{k_2}-\varepsilon_{k_1})t}\, u^*_{k_1}(r)\, u_{k_2}(r)$$

$$= i \int dr \int dk_1 dk_2 \; \omega^*_{k_1-k_0} \, \omega_{k_2-k_0} \, e^{i(k_1-k_2)r}$$

$$\left(-it \frac{\partial \varepsilon_{k_1}}{\partial k_1} u^*_{k_1}(r) u_{k_2}(r) + \frac{\partial u^*_{k_1}(r)}{\partial k_1} u_{k_2}(r) \right) e^{i(\varepsilon_{k_1}-\varepsilon_{k_2})}$$

$$= \int dk_1 dk_2 \; \delta(k_1-k_2) \, \omega^*_{k_1-k_0} \, \omega_{k_1-k_0}$$

$$\left(-it v_{k_1} \underbrace{\langle u^*_{k_1} u_{k_1} \rangle}_{=1} + i \langle \frac{\partial u^*}{\partial k_1} | u_{k_1} \rangle \right) \underbrace{e^{i(\varepsilon_{k_1}-\varepsilon_{k_1})}}_{1}$$

$$= \int dk_1 \underbrace{\omega^*_{k_1-k_0} \, \omega_{k_1-k_0}}_{\simeq \delta\text{-function}} \left(-it v_{k_1} + i \langle \frac{\partial u^*}{\partial k_1} | u_{k_1} \rangle \right) \simeq$$

$$\simeq v_{k_0} t + i \langle \frac{\partial u^*}{\partial k} | u_k \rangle_{k=k_0}$$

~~motion with velocity~~

$$\cancel{v_k = \frac{\partial \varepsilon}{\partial k} + \langle \frac{\partial u}{\partial k} | u_k \rangle}$$

$$r_c(k_0,t) = v_{k_0} t + \underbrace{\delta r_{-\infty}}_{= \langle u_{k_0} | i \frac{\partial}{\partial k} u_{k_0} \rangle}$$

The scattering of the wave packet on the static impurity

$V(r)$

$$\Psi_k^{out}(r,t) = \int dk'\, C(k',t)\, \Psi_{k'}(r,t)$$

In the lowest order

$$C(k',t) = -\underbrace{\langle k|V|k'\rangle}_{T_{k,k'}} \int_{-\infty}^{t} e^{i(\varepsilon_{k'}-\varepsilon_k)t' - \varepsilon|t'|}\, dt' + \delta(k'-k)$$

$$= -2\pi i\, T_{k',k}\, \delta(\varepsilon_k - \varepsilon_{k'}) + \delta(k-k')$$

$$\Psi^{out}(r,t) = \int dk\, \omega(k-k_0)\, \Psi_k^{out}(r,t)$$

wave packet

$$\underset{t\to-\infty}{r_c(t)} = \int dr \left(\Psi^{out\,*}(r,t)\, r\, \Psi^{out}(r,t) \right) =$$

$$= \int dk' \left|C(k',\infty)\right|^2 \left(v_{k'} t + \langle u_k | i \tfrac{\partial}{\partial k'} u_{k'} \rangle - \left(\tfrac{\partial}{\partial k} + \tfrac{\partial}{\partial k'} \right) \mathrm{Arg}\left(\langle k|V|k'\rangle \right) \right)$$

$$r_c(t) \underset{t\to\infty}{\widehat{=}} \sum_{k'} W_{k_0,k'} \Big(v_{k'} t + \langle u_{k'} | i \tfrac{\partial}{\partial k'} u(k') \rangle$$

$$- \left(\frac{\partial}{\partial k} + \frac{\partial}{\partial k'} \right) \mathrm{Arg} \left(V_{k,k'} \right) \Big)$$

$$r_c(t) \underset{t\to-\infty}{=} v_{k_0} t + \langle u_{k_0} | i \frac{\partial}{\partial k_0} | u_{k_0} \rangle$$

$$\delta r_{kk'} = \langle u_{k'} | i \frac{\partial}{\partial k'} | u_{k'} \rangle - \langle u_k | i \frac{\partial}{\partial k} u_k \rangle -$$

$$- \left(\frac{\partial}{\partial k} + \frac{\partial}{\partial k'} \right) \mathrm{Arg}\, V_{k,k'}$$

$$r_c(t) \underset{t\to\infty}{=} \sum_{k'} \underbrace{W_{k_0,k'}}_{\text{probability to scatter from } k_0 \to k'} \Big(\underbrace{r_c(t=-\infty)}_{\text{initial coordinate}} + \underbrace{\delta r_{k_0,k'}}_{\text{coordinate shift}} \Big)$$

Anomalous Hall effect in Weyl semimetal.

Focus on 3D Hamiltonian

$$H = u_{\parallel}(p_x \sigma_x + p_y \sigma_y) + \left(\frac{p_z^2}{2m} - \lambda\right)\sigma_z + V$$

This is Weyl semimetal with TR symmetry broken

Linear respondse Boltzmann eq.

$$\frac{\partial f}{\partial t}(l) + e\vec{E}\nabla_l f = -\sum' W_{ll'}\left[f_l - f_{l'} - \frac{\partial f}{\partial \epsilon} e\vec{E}\delta r_{l'l}\right]$$

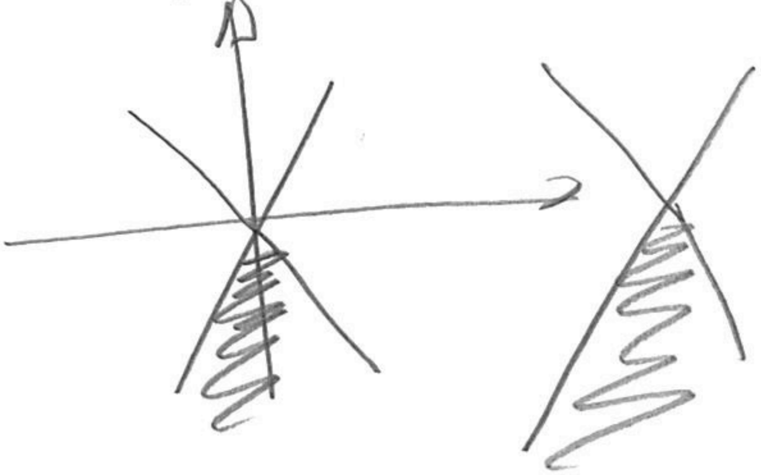
$$l = |u_\alpha, k\rangle \qquad l' = |u_{\alpha'}, k'\rangle$$

$$v_l = \nabla_k \epsilon_k + \frac{dp}{dt} \times \vec{F}_l + \sum_{l'} \delta r_{l'l} W_{l'l}$$

$$F_l = -i\langle \nabla_k u_{\alpha k}| \times |\nabla_k u_{\alpha k}\rangle$$

$$\delta r_{l'l} = \langle u_{l'}|i\nabla_k u_{l'}\rangle - \langle u_l|i\nabla_k u_l\rangle -$$
$$- (\nabla_{k'} + \nabla_k)\arg\langle u_{l'}|u_l\rangle$$

At the Weyl node the only contribution to current



$j = e\sum_{l} f_l v_l$ is given by anomalous term

$$j = e\sum_{l} f_l \left(\nabla \mathcal{E}_l + \vec{E} \times \vec{F} + \delta r_{l'l} w_{l'l} \right) =$$

$$= e\sum_{l} f_l \left(\vec{E} \times \vec{F} \right)$$

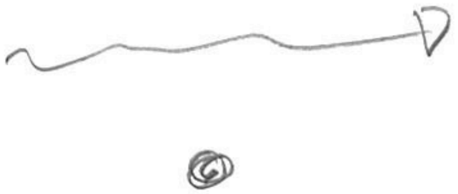
$$\sigma_{xy}^{int} = \sum_{l} f_l \left(\vec{F}_l \right)_z \overset{\mu=0}{=} \frac{e^2}{2h} \underbrace{\Delta k}_{\text{distance between Weyl nodes}}$$

This is "quantised" anomalous Hall conductivity

<u>intrinsic</u>

In addition, for chemical potential that is not exactly at the Weyl node there are other terms.

If the wave length is bigger that the size of the imurity [slow electrons]



we can use s-wave approximation.
More carefully, becaus states are spinors we have two scattering phases δ_+ and δ_- for states with total angular momentum $J = \pm 1/2$

$$tg\,\delta_\pm = -\pi\, T_\pm\, \nu_\pm(\epsilon)$$

density of states for $\pm$ components

T-matrix

Can be computed exactly!
see our coming paper

The Hall resistivity

skew scattering

$$\rho_{xy} = -\, n_{imp} \frac{\sin\delta_+ \sin\delta_-}{\pi e^2 \nu_+ \nu_-} \sin(\delta_+ - \delta_-) - \rho_{xx}^2 \sigma_{xy}^{int}$$

$\boxed{\text{if } \delta_+ = \delta_- \text{ back to intrinsic}}$

$$\hat{\rho} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \qquad \hat{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}$$

$$\hat{\rho}^{-1} = \hat{\sigma}$$

$$\hat{\rho} = \hat{\sigma}^{-1} = \frac{1}{\sigma_{xx}\sigma_{yy} - \sigma_{xy}\sigma_{yx}} \begin{pmatrix} \sigma_{yy} & -\sigma_{xy} \\ -\sigma_{yx} & \sigma_{xx} \end{pmatrix}$$

$$\simeq \frac{1}{\sigma_{xx}^2} \begin{pmatrix} \sigma_{yy} & -\sigma_{xy} \\ \sigma_{xy} & \sigma_{xx} \end{pmatrix}$$

$$\hat{\rho}_{xy} \simeq -\frac{\sigma_{xy}}{\sigma_{xx}^2} = -\rho_{xx}^2 \sigma_{xy}$$

$$\sigma_{inf} \simeq n_{imp} \frac{\sin\delta_+ \sin\delta_-}{\pi e^2 \nu_+ \nu_-} \sin(\delta_+ - \delta_-)\, \sigma_{xx}^2 =$$

$$\cong \left(e^2 \nu D_{xx}\right)^2 \frac{n_{imp} \sin\delta_+ \sin\delta_-}{\pi e^2 \nu^2} \sin(\delta_+ - \delta_-) \simeq$$

$$\simeq e^2 n_{imp} D^2 \sin\delta_+ \sin\delta_- \sin(\delta_+ - \delta_-)$$

In the ~~second~~ lowest Born approximation $\delta_+ = \delta_-$ and skew scattering vanishes

However, for beyond Born approximation it is important. Away from the neutrality ~~For any finite size σ_{xy}~~ point, sufficiently clean semi-metals ~~are dominate~~, σ_{xy} is determined by skew scattering