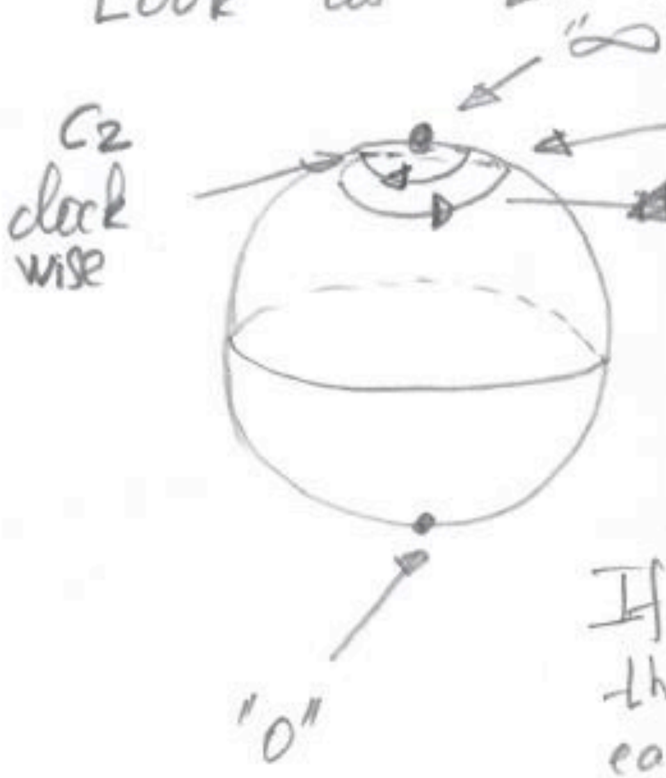
We understoud that point at infinity can be treated like any other point.

From here we can naturaly define the Res $f(\infty)$.

Look at the Riemann sphere



two parallel circles

counter clock wise C_1

$$\oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz = 0$$

If two circles are closed than integrals cancell each other.

Now we pull the countour C_2 down and by Cauchy's theorem it becomes

$$\oint_{C_2} f(z)\,dz = 2\pi i \sum_n \operatorname{Re} f(z_n)$$

all residue at point on the complex plane (Riemann sphere) except ∞ (N pole)

We got

$$\oint_{C_2 \text{ clockwise}} f(z)dz + 2\pi i \sum_n \operatorname{Re} f(z_n) = 0$$

Now define $\operatorname{Res} f(\infty) = \frac{1}{2\pi i} \oint_{C_2} f(z) dz$

clockwise around the point ∞ (N pole)

Sum of Residues of analytic function is zero

$$\boxed{\sum_n f(z_n) + f(\infty) = 0}$$

Important, we assumed that the function has only pole singularities on the complex plane.

Les us check how it works.

Consider example.

$$f(z) = \frac{1}{z}$$

$$\operatorname{Res} f(0) = \frac{1}{2\pi i} \oint_{C_1} \frac{1}{z}\, dz =$$

around zero
anticlockwise

$z = \varepsilon e^{i\varphi}$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{\cancel{\varepsilon e^{i\varphi}}}\, i d\varphi\, \cancel{\varepsilon e^{i\varphi}} = 1$$

$$\operatorname{Res} f(\infty) = \frac{1}{2\pi i} \oint_{C_2} \frac{1}{z}\, dz =$$

around ∞, clockwise

$z = Re^{i\varphi} \qquad R \gg 1$

$$= \frac{1}{2\pi i} \int_0^{-2\pi} \frac{1}{\cancel{Re^{i\varphi}}}\, i d\varphi\, \cancel{Re^{i\varphi}} = -\frac{2\pi i}{2\pi i} = -1$$

$\operatorname{Res} f(0) + \operatorname{Res} f(\infty) = 0$ ✓

In a slightly different way the same example.
Consider an integral around ∞

$$\operatorname{Re} f(\infty) = \frac{1}{2\pi i} \oint_{C_2 \text{ clockwise}} \frac{1}{z}\, dz$$

and perform a mapping $z = \frac{1}{\zeta}$

As we know this is Möbius mapping that rotates Riemann sphere, sending $\infty \to 0$

So we can understan the integral around the point ∞ in several equivalent ways.

1) As an integral in clockwise direction on a large circle on the complex plane [small circle around N pole on the Riemann sphere]

2) We can apply a Möbius transformation $\zeta = 1/z$ sending $\infty \to N$ and clockwise contur in z plane to anticlockwise countur in ζ plane.

In this way

$$\oint_{\substack{\text{clockwise}\\ \text{circle around}\\ \infty}} dz\, f(z) = \oint_{\substack{\text{anticlockwise}\\ \text{circle around}\\ 0}} \left(-\frac{d\zeta}{\zeta^2} f(\zeta)\right)$$

For our example $\boxed{2\pi i \operatorname{Re} z(\infty) =}$

$$= \oint_{\substack{\text{clockwise}\\ \text{around}\\ \infty}} \frac{dz}{z} = \oint_{\substack{\text{anticlockwise}\\ \text{around } 0}} -\frac{d\zeta}{\zeta^2}\,\zeta = -\oint_{\text{anticlockwise}} \frac{d\zeta}{\zeta} = -2\pi i \Rightarrow \boxed{\operatorname{Res} z(\infty) = \underline{-1}}$$

and $\operatorname{Re} z(0) + \operatorname{Re} z(\infty) = 1 - 1 = 0$ as expected

$$\mathrm{Res}\, f(\infty) = \frac{1}{2\pi i} \oint_{C_1} z \, d\frac{1}{z} = \frac{-1}{2\pi i} \oint_{C_1} \frac{dz}{z} = -1$$

(anticlock wise around 0)

Now we understand how complex variables help to compute ordinary integrals. We now discuss how they also help in solving partial differential eqs.

For example Laplace equation.

$\Delta \varphi = 0$. Here $\Delta \equiv \nabla^2 = \mathrm{div} \cdot \mathrm{grad}$

This equation appears frequently in physics, for example in electrostatics where Poisson equation for electric potential

$$\Delta \varphi = 4\pi \rho$$

ρ — density of electric charge

φ — electrostatic potential

if $\rho = 0$ in some area Poisson eq. reduces to Laplace eq.

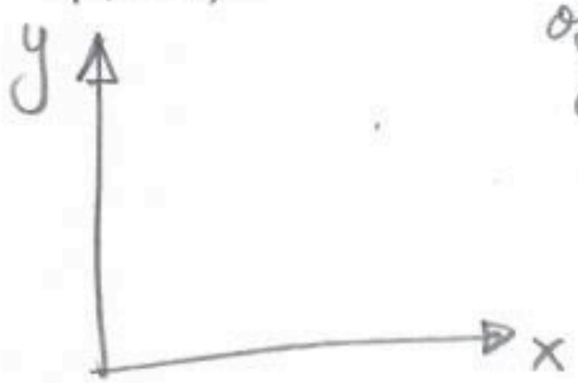
Laplace eq. also appears in hydrodynamics and other ares [to be discussed in the future]

Any Functions that satisfy the Laplace equation are called <u>harmonic</u> functions

We can study the harmonic functions in 2D via complex analysis.

First, let us combite the coordinates x, y of the real 2D plane into a complex variable

$$z = x + iy$$



Then we can conect the harmonic function in 2D with analytic functions on the complex plane z.

<u>Theorem</u>

If $f(z) = u(x,y) + i\,v(x,y)$ is analytic in the domain D of the complex plane then both $u(x,y)$ and $v(x,y)$ are harmonic functions on D.

Proof is based on Caushy-Riemann eqs.

$$\frac{\partial}{\partial x} u(x,y) = \frac{\partial}{\partial y} v(x,y)$$

$$\frac{\partial u(x,y)}{\partial y} = -\frac{\partial}{\partial x} v(x,y)$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x \partial y} v(x,y)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2}{\partial x \partial y} v(x,y)$$

[here we assume that $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ are interchangable]

Add together we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Similarly for $v(x,y)$ $\blacksquare$

This theorem works in the opposite direction as well.

If $u(x,y)$ is harmonic on simply connected domain D, then $u(x,y) = \text{Re } f(z)$

$f(z) = u(x,y) + i v(x,y)$ is an analytic function on D.

[Proof to be read in textbook]

If harmonic functions u and v are real and imaginary parts of an analytic function, then we call them harmonic conjugates.

Note that if

$f(z) = u + iv$ is analytic so is

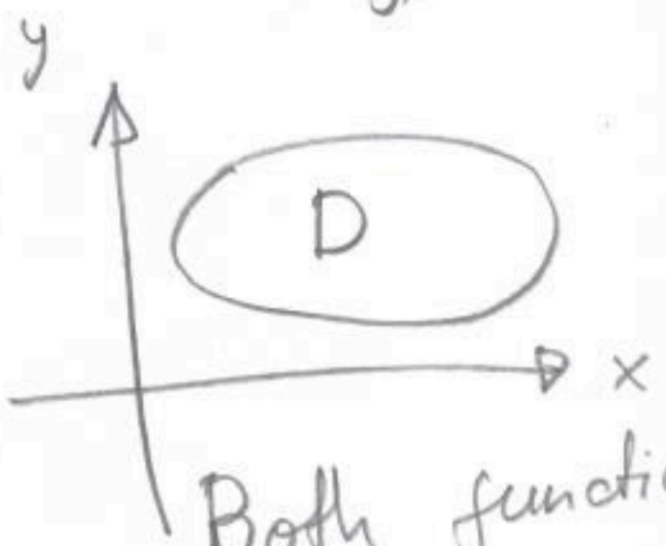
$if(z) = -v + iu$

therefore u and $-v$ are also harmonic conjugates.

Example:

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} =$$

$$= \underbrace{\frac{x}{x^2+y^2}}_{u(x,y)} - i \underbrace{\frac{y}{x^2+y^2}}_{v(x,y)}$$



domain D does not contain the origin

Both functions $u(x,y) = \frac{x}{x^2+y^2}$ and

$v(x,y) = \frac{-y}{x^2+y^2}$ are

harmonic functions. They are harmonic conjugates.

Theorem

The gradients of u and v are orthogonal

$$(\nabla u)\cdot(\nabla v) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)\cdot\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) =$$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} =$$ Cauchy-Riemann eqs.

$$= \left(\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right) - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0$$

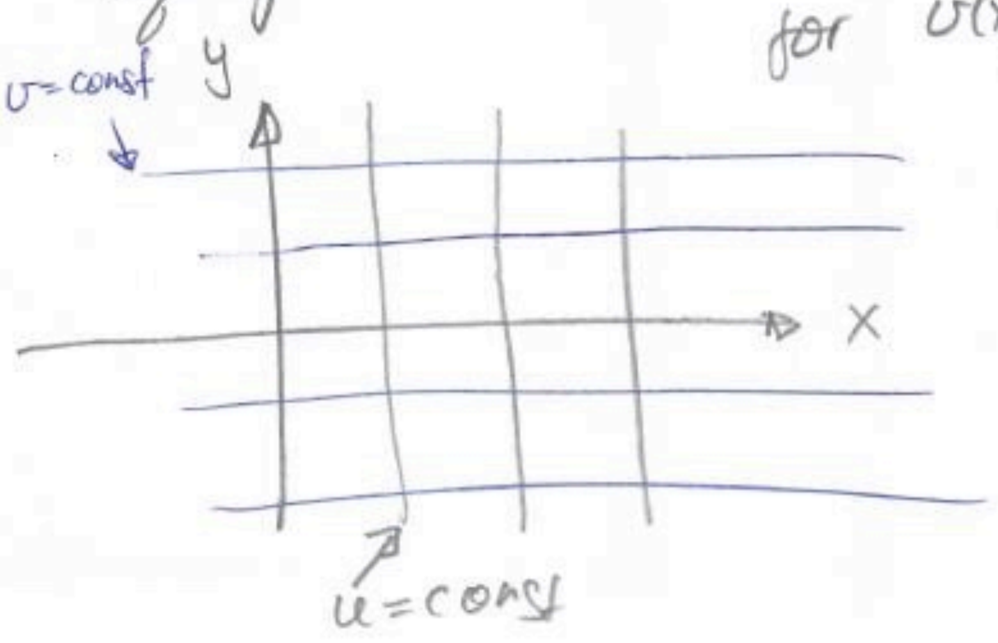
The consequence of this, is that if levels of the function $u(x,y)$ and $v(x,y)$ [i.e. equapotential lines] are smooth, they are orthogonal.

Examples

$$f(z) = z = \underset{u}{x} + iy \quad y = v$$

Equapotential lines for $u(x,y) \Rightarrow x = \text{const}$.

for $v(x,y) = y = \text{const}$



First branch of ln on the complex plane with a cut

$$f(z) = \ln z = \ln R + i\varphi$$

$$z = Re^{i\varphi}$$

$$u = \operatorname{Re} f = \ln R$$

$$v = \operatorname{Im} f = \varphi$$