Lecture 1

Consider the equation

$$z^2 = -1$$

It has no roots for $z \in$ Real

However formally

$$z = \pm\sqrt{-1}$$

$$i \equiv \sqrt{-1}$$

$$z = \pm i$$

Obviously $i^2 = -1$

Define the complex number as any number of the form

$$z = x + iy$$

$x, y \in$ Real

Example

$3+2i$ is a complex number

Real part of the complex number

$Re(z) = Re(x+iy) = x$

$Re(3+2i) = 3$

Imaginary part of the complex number

$Im(x+iy) = y$

$Im(3+2i) = 2$

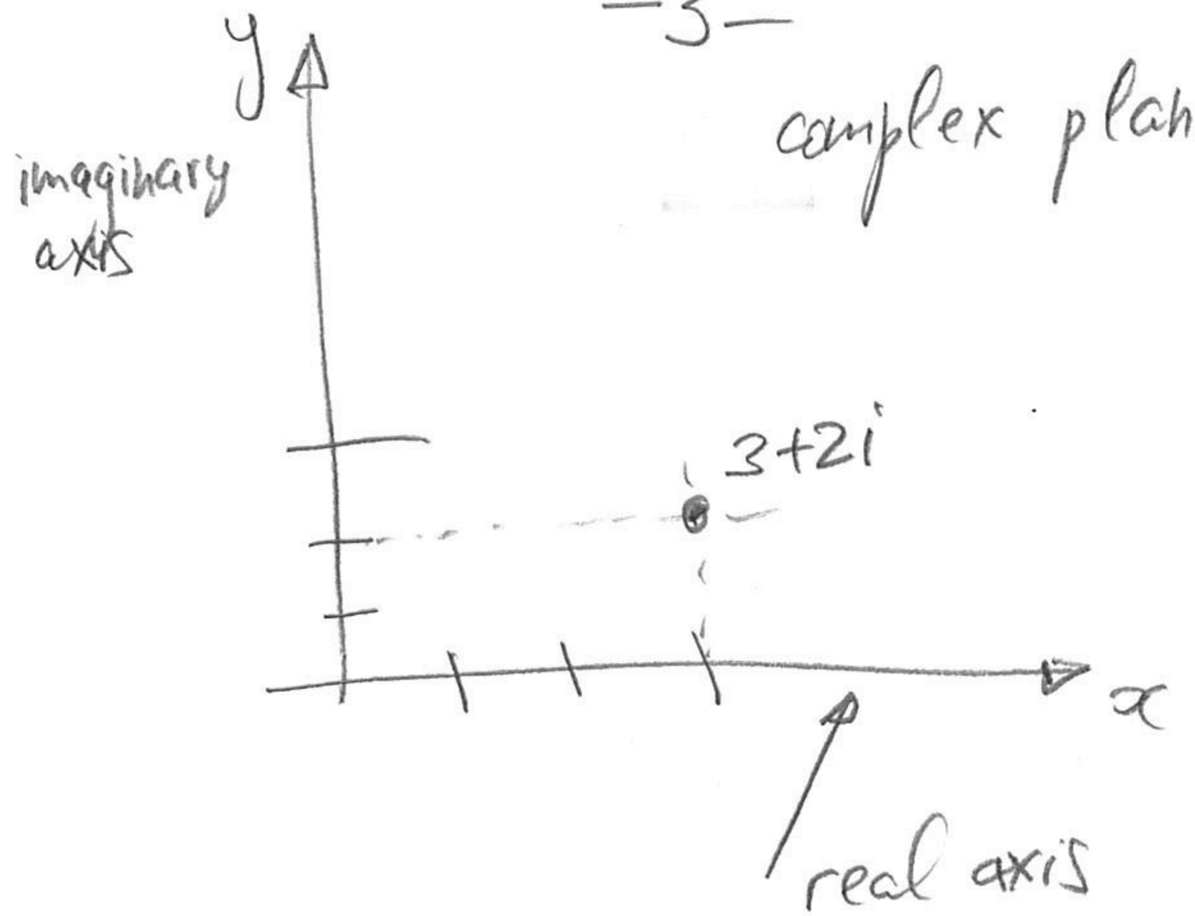
The case $y=0$

$z = x + i \cdot 0 = x$ is a real number

The case $x = 0$

$z = iy$ is a purely imaginary

Note that we can visualize the pair (x,y) as a point on a plane complex plane



Two complex number $z_1 = z_2$

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

iff $x_1 = x_2$ and $y_1 = y_2$

Addition of complex numbers

$z = z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = x_1 + x_2 + i(y_1 + y_2)$

$\mathrm{Re}\, z = \mathrm{Re}\, z_1 + \mathrm{Re}\, z_2$

$\mathrm{Im}\, z = \mathrm{Im}\, z_1 + \mathrm{Im}\, z_2$

Example

$z_1 = 3 + 2i$

$z_2 = -1 - i$

$z = z_1 + z_2 = 3 - 1 + (2-1)i = 2 + i$

Obviously

$$z_1 + z_2 = z_2 + z_1$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

Multiplication

$$z = z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) =$$

$$= x_1 x_2 + i(y_2 x_1 + x_2 y_1) + iy_1 \cdot iy_2 =$$

$$= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$$

Obviously

$$z_1 \cdot z_2 = z_2 z_1$$

$$(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$$

Example

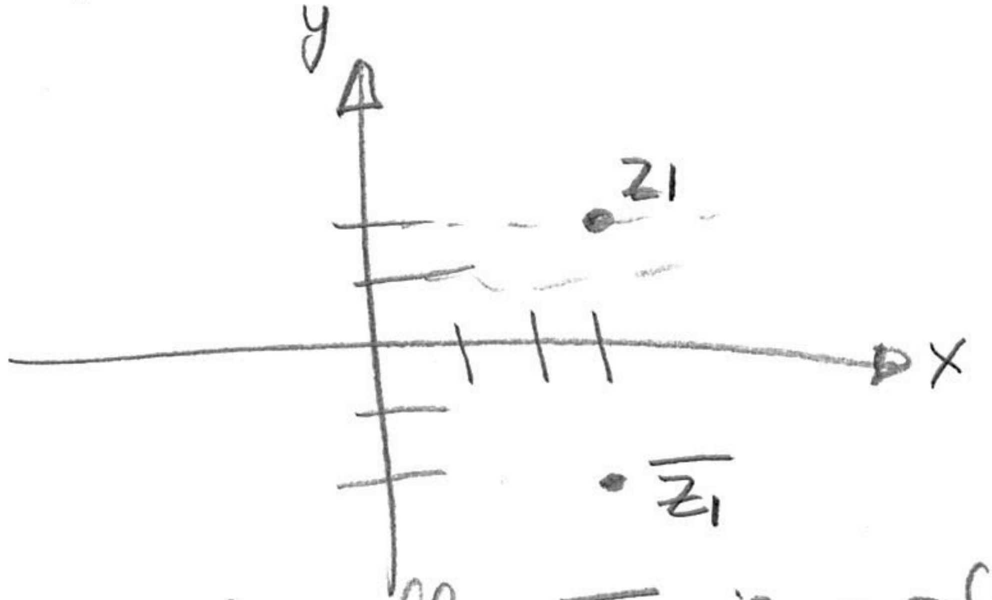
$$z_1 \cdot z_2 = (3+2i)(-1-i) = -3-3i-2i+2$$

$$= -1-5i$$

complex conjugation

$$\overline{z} = z^* = \overline{(x+iy)} = x-iy$$

$$\overline{z_1} = \overline{3+2i} = 3-2i$$



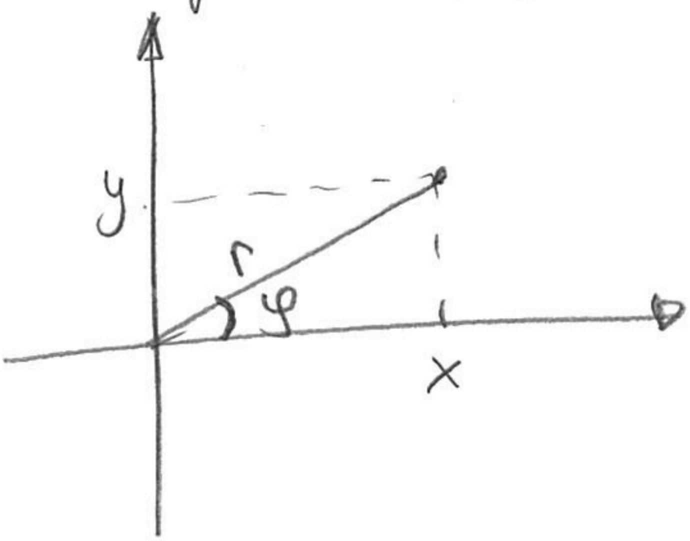
geometrically $\overline{z}$ is a reflection of a point with respect to the real axis

$$z^2 = z \cdot z$$

$$z^3 = z \cdot z \cdot z$$

$$z^n = \underbrace{z \cdot z \cdot z \ldots z}_{n \text{ times}}$$

As we saw above the complex number is a point on a plate



$$r = \sqrt{x^2 + y^2}$$

$$tg\,\varphi = \frac{y}{x}$$

$$\varphi = arctg(y/x)$$

$$z = x + iy = r\cos\varphi + i r \sin\varphi =$$

$$= r\underbrace{(\cos\varphi + i\sin\varphi)}_{e^{i\varphi}} = r e^{i\varphi}$$

(to be shown)

$$z_1 \cdot z_2 = r_1(\cos\varphi_1 + i\sin\varphi_1)\, r_2(\cos\varphi_2 + i\sin\varphi_2)$$

$$= r_1 r_2 \left(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i\left[\cos\varphi_2\sin\varphi_1 + \cos\varphi_1\sin\varphi_2\right]\right) =$$

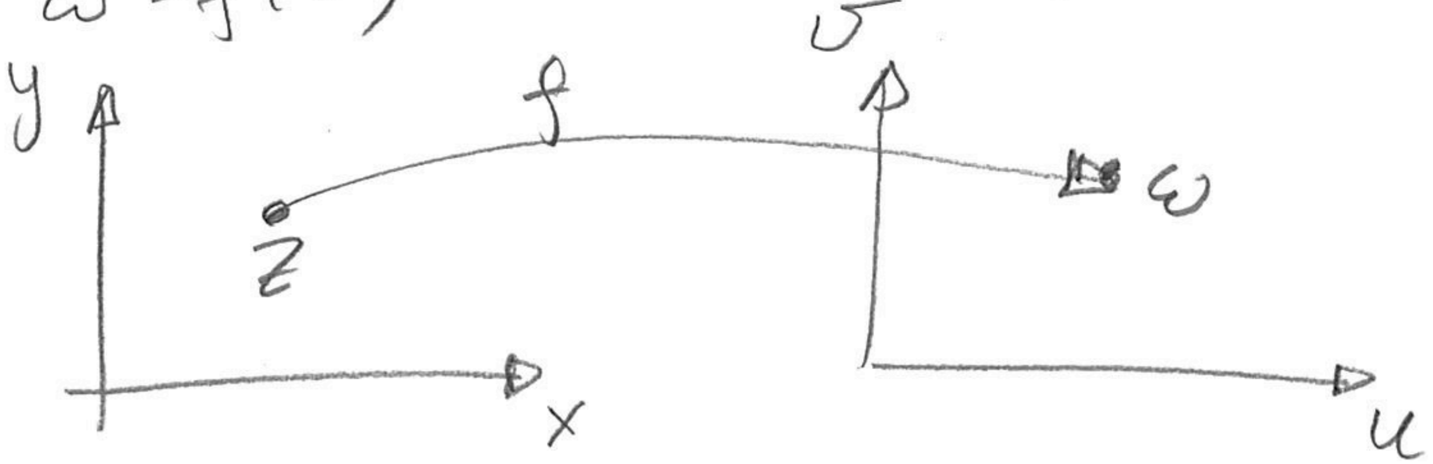
$$= r_1 r_2 \left(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)\right) =$$

$$= r_1 r_2\, e^{i(\varphi_1+\varphi_2)}$$

Functions of complex variable

$z = x + iy$ $\qquad \omega = u + iv$

$\omega = f(z)$ means $u(x,y)$ and $v(x,y)$



Example

$\omega = z^2 \qquad f(z) = z^2$

$z = x + iy$

$\omega = (x+iy)^2 = x^2 - y^2 + 2ixy$

$u = x^2 - y^2$

$v = 2xy$

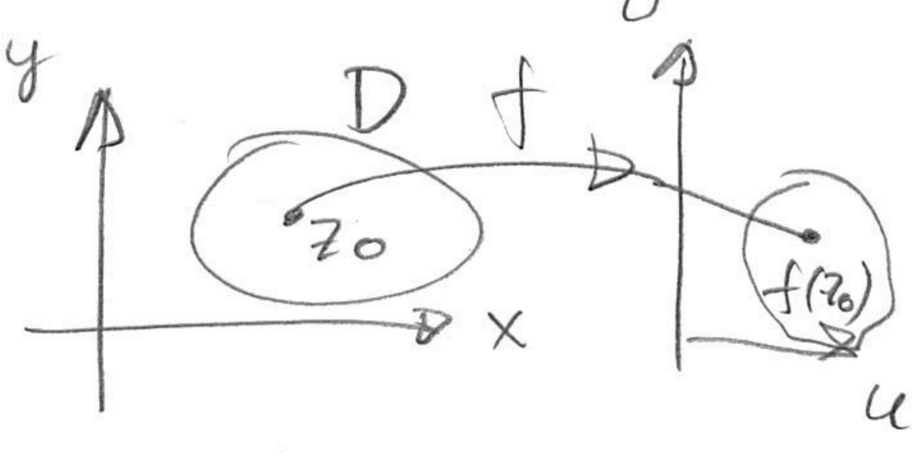
Derivatives, limits, continuity

$$\lim_{z \to z_0} f(z) = f(z_0) = u_0 + i v_0 = \omega_0$$

For continuous function

$$\lim_{z \to z_0} f(z) = f(z_0)$$

If the function is continous in the region D



The derivative of the function

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

note that in the complex plane there different ways to appoach the point from different direction.

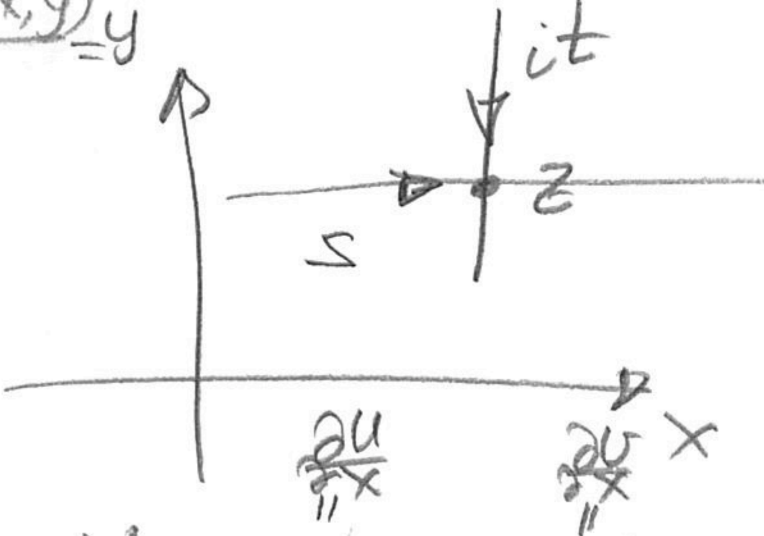
The derivative $f'(z)$ does not depend on the way. For this, to be the case the following condition must be satified

Cauchy-Riemann equations

$$\boxed{\begin{aligned} \frac{\partial u(x,y)}{\partial x} &= \frac{\partial v(x,y)}{\partial y} \\ \frac{\partial u(x,y)}{\partial y} &= -\frac{\partial v(x,y)}{\partial x} \end{aligned}}$$

Let us prove this. Let us first follow the real axis $h=s$

$$f'(z) = \lim_{s\to 0} \frac{f(x+s,y)-f(x,y)}{s}$$



$$= \lim_{s\to 0} \left\{ \frac{u(x+s,y)-u(x,y)}{s} + i\,\frac{v(x+s,y)-v(x,y)}{s} \right\} = u_x(x,y) + i v_x(x,y)$$

NOW $h = it$ (go along the vertical line)

$$f'(z) = \lim_{s\to 0} \left\{ \frac{u(x,y+t)-u(x,t)}{it} + i\,\frac{v(x,y+t)-u}{it} \right\}$$

$$= \frac{\partial u}{i\,\partial y} + i\,\frac{\partial v}{i\,\partial y} = -i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

demand the real and imaginary parts are equal

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}}$$