Following Jeremy Orloff MIT course

Lecture 10

# Conformal transformations

Conformal transformations are functions on the complex plane that preserve the angles between curves.

Suppose $\gamma(t) = (x(t), y(t))$ is a curve on the plane. $f(z)$ is differentiable function of $z$ at $z_0$. $z = x+iy$ and $\gamma(t_0) = z_0$

The map $f(\gamma) \equiv \tilde{\gamma}(t)$ is a new map.

The tangent vector of $\gamma(t)$ is

$$\frac{d\gamma(t)}{dt} = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right)$$

The tangent vector of the new map $\tilde{\gamma}$

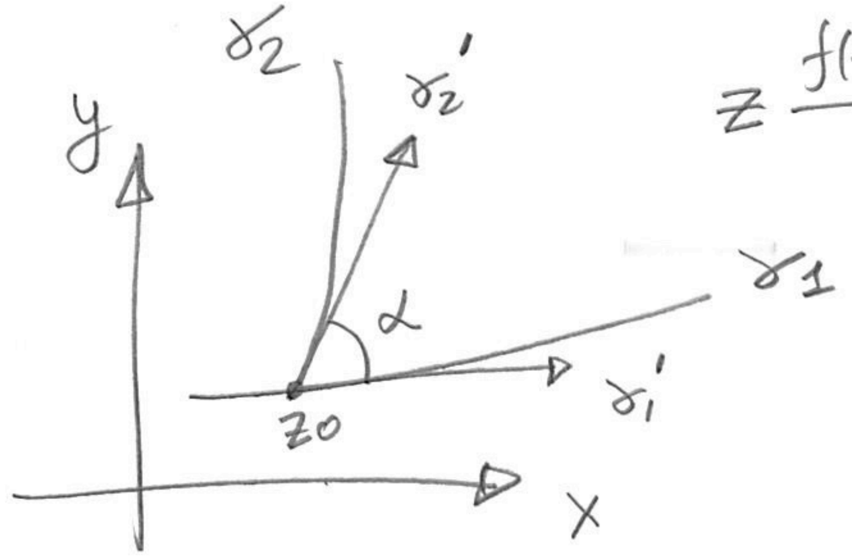
$$\frac{d\tilde{\gamma}}{dt} = \frac{d}{dt} f(\gamma(t)) = \frac{df}{dz}\frac{d\gamma}{dt} =$$

$$= \frac{df}{dz}\left(\frac{dx}{dt} + i\frac{dy}{dt}\right)$$

The components of the tangent vector are

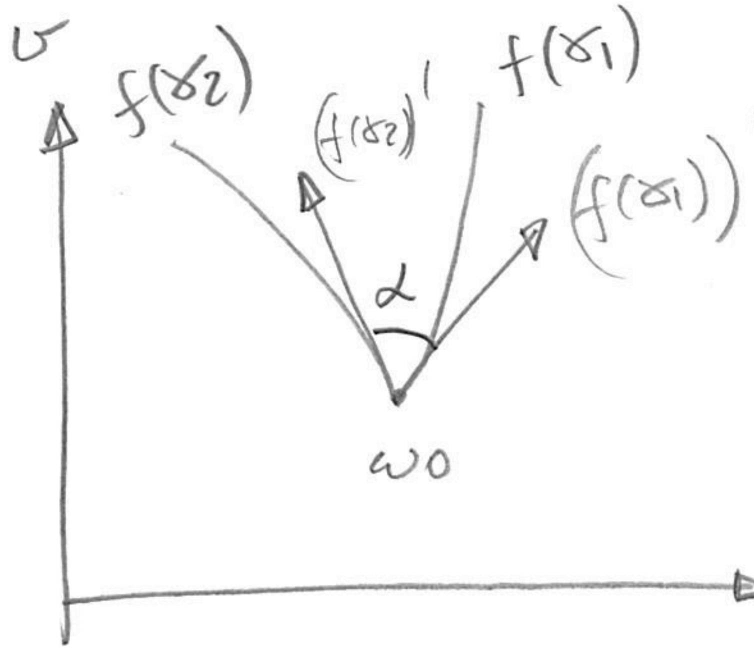
$$\left(\frac{df}{dz}\frac{dx}{dt}, \frac{df}{dz}\frac{dy}{dt}\right)$$

$z \xrightarrow{f(z)} \omega$




$z = x + iy$

$\omega = u + iv$

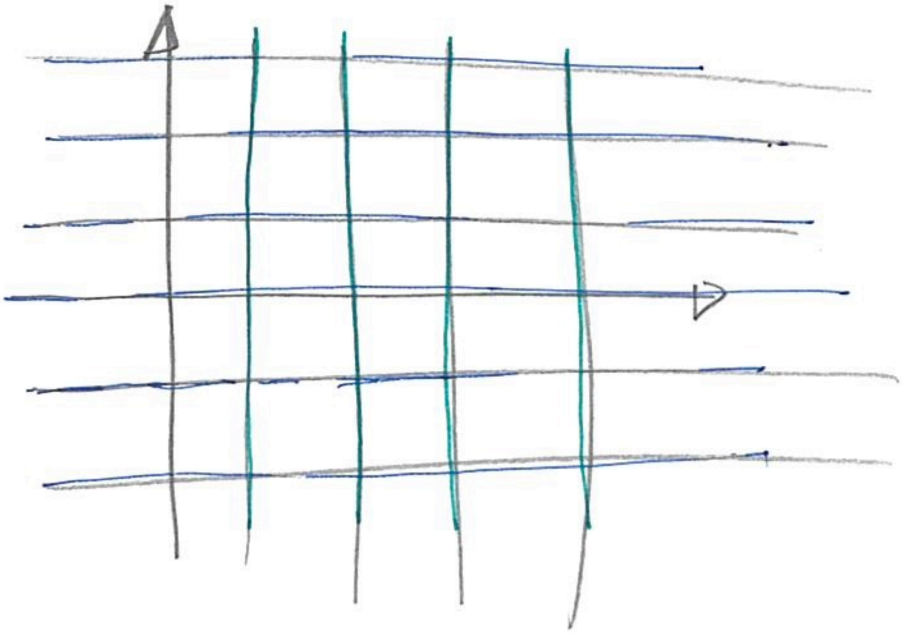



The function $f(z)$ is conformal at $z$ if tangent vector of any smoth curve at $z$ is rotated by the same angle and scaled by the same scalar.

# Example of conformal map

$f(z) = z^2$

Look on the grid



$z = x + ih$

$$f(z) = z^2 = (x+ih)^2 = x^2 + 2ixh - h^2 =$$
$$= x^2 - h^2 + 2ihx$$

$u = x^2 - h^2$

$v = 2hx$

$$u = (2hx)^2 \frac{1}{4h^2} - h^2 = \frac{v^2}{4h^2} - h^2$$

blue lines

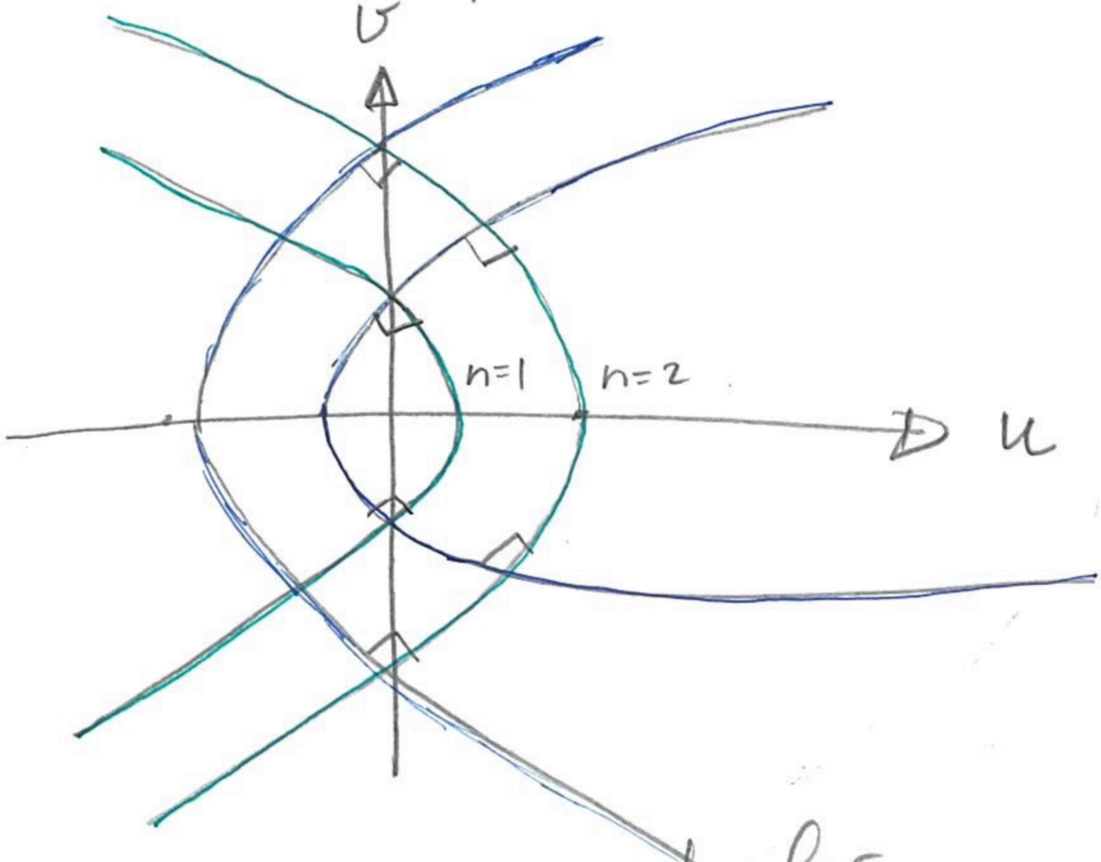
$z = x+iy$ green lines

$$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

$$u = x^2 - y^2$$

$$v = 2xy$$

$$u = x^2 - \frac{(2xy)^2}{4x^2} = x^2 - \frac{v^2}{4x^2}$$




grid lines remain perpendicular

Tangent vectors

$\gamma'(t) = (x'(t), y'(t)) \longleftrightarrow x'(t) + i y'(t)$

compex representation of a vector

Thus tangent vector in plane is a complex number

Therefore the conformal transformation that rotates and scales the vector is just a multiplication by a complex number $|a|e^{i\phi}$ rotation angle

scale

All analytic functions are conformal.

Proof. $\gamma(t) \xrightarrow{f} f(\gamma(t))$

$$\frac{d}{dt} f(\gamma(t))\Big|_{t=t_0} = f'(\gamma(t_0))\,\gamma'(t_0) = f'(z_0)\gamma'(t_0)$$

but $f'(z_0)$ is a complex number

$f'(z_0) = |f'(z_0)|e^{i\,\mathrm{Arg} f'(z_0)}$ therefore

it is conformal.

Simplest example of conformal map is a constant map

$$f(z) = c \cdot z$$

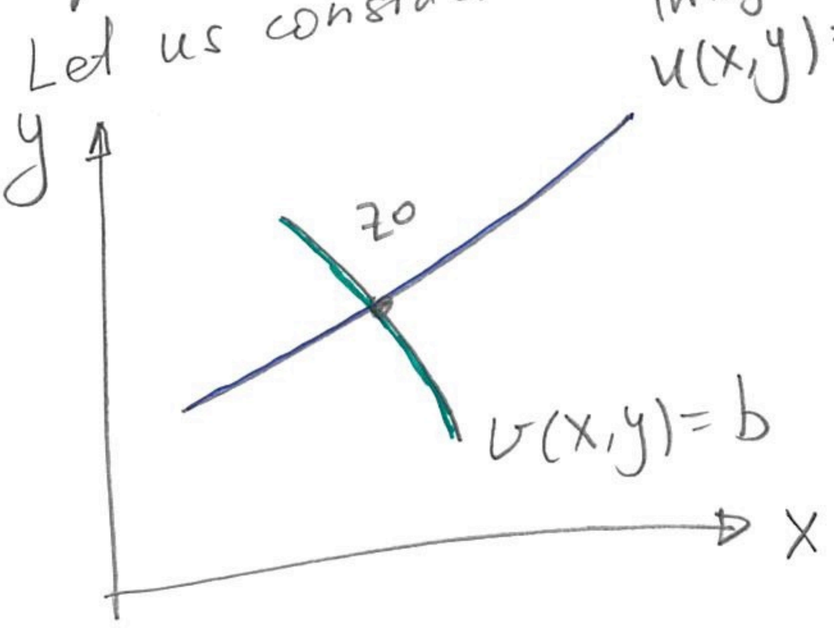
## Digression to harmonic functions.

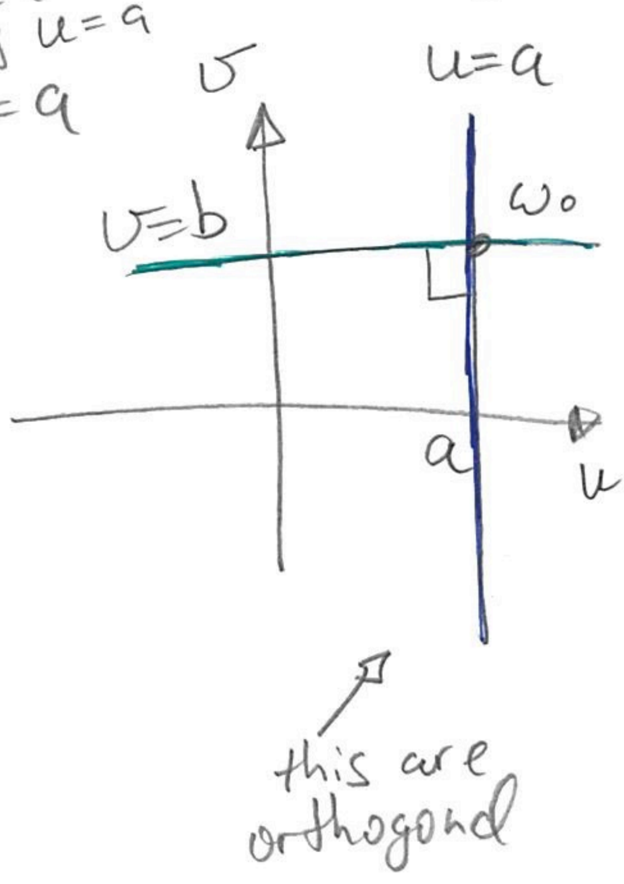
If $u$ and $v$ are harmonic conjugates and $g = u + iv$ has $g'(z_0) \neq 0$ then the levels of $u$ and $v$ are orthogonal.

We have already shown this via Cauchy-Riemann equations. Here we ~~repeat. this theo~~ rederive this result using the idea of conformal maps.

Let us consider the level line of $u(x,y)=a$

image of $u=a$




$$g(z_0) = w_0$$

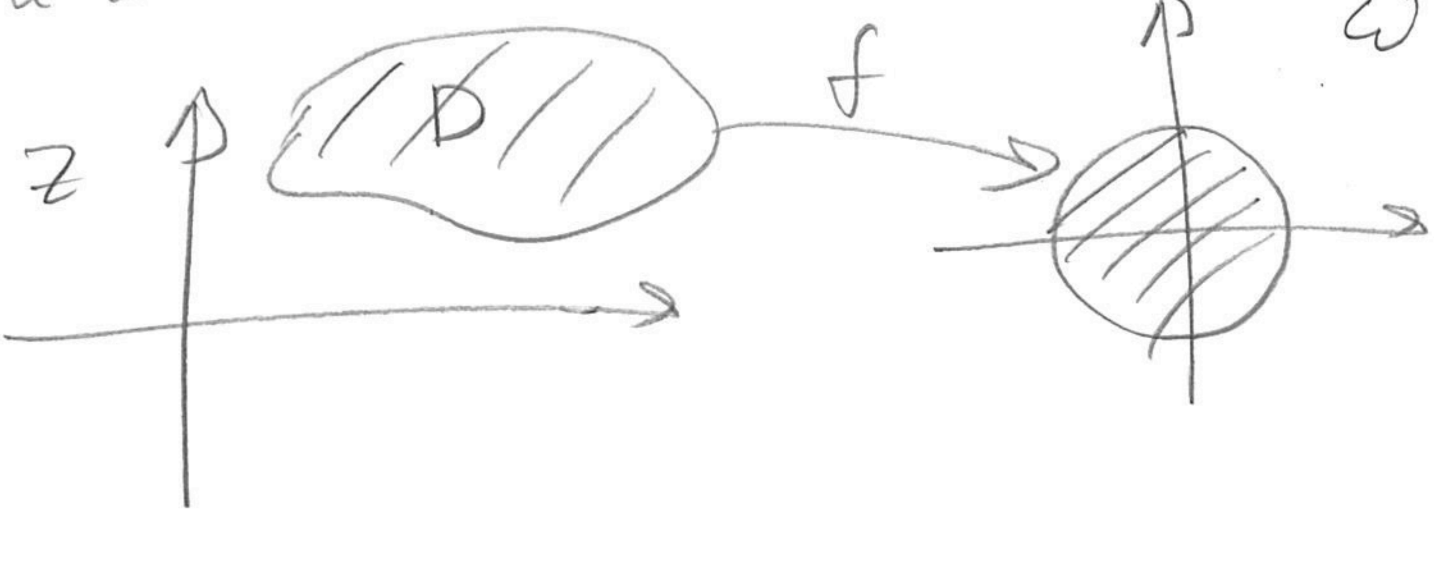
Sincle the map $g(x,y)$ is conformal the angle between the curves is preserved. Therefore the level lines are orthogonal.

## Riemann mapping theorem

[very fundamental in math.]

Any two simple connected domains [other than the entire plane] on the complex plane can be conformally maped to each other.

For example, any simply connected domain [other than the entire plane] can be conformally mapped onto a unit disk



Fractional linear transformations

$$T(z) = \frac{az+b}{cz+d}$$

These are the Möbius transformations we discussed in the context of Riemann sphere. We have interpreted them as rotations of the sphere. Naturally, the angles were preserved. That is why they are globally conformall (on the entire plane).

Let us look at one particular example.

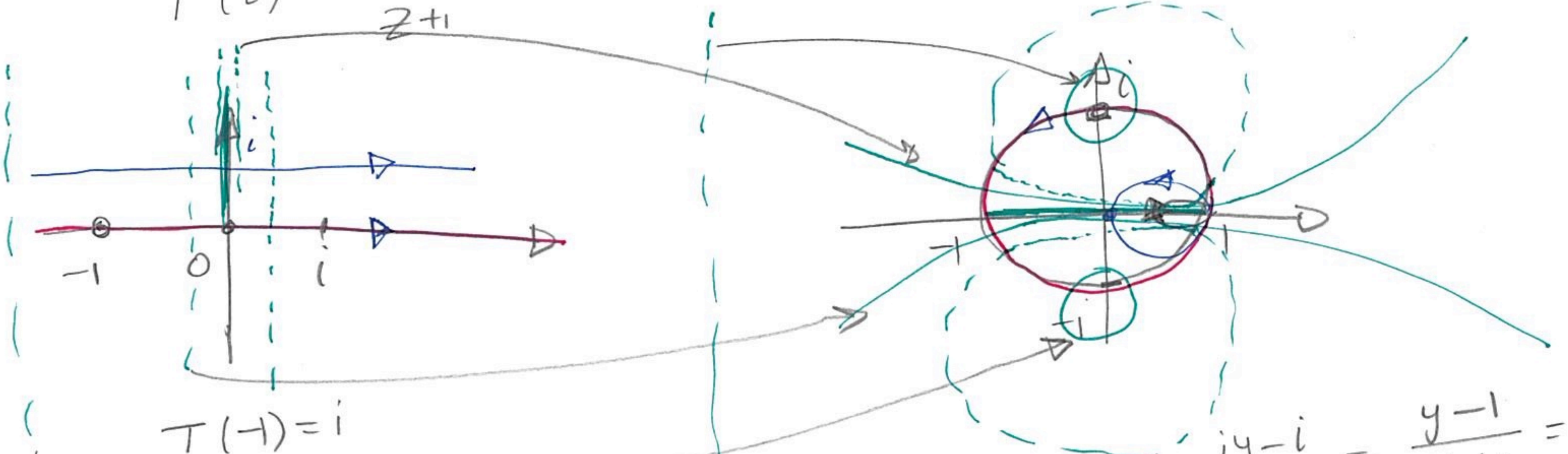
$$T(z) = \frac{z-i}{z+i}$$

and draw it in details. We show that it maps conformaly the upper half plane to the interier of a unit circle (unit disk).

~~the half plane is mapped to~~

# half plane to the unit circle

$$T(z) = \frac{z-i}{z+i}$$




$T(-1) = i$

$T(0) = -1$

$T(1) = -i$

Line $\quad z = iy \longrightarrow \dfrac{iy-i}{iy+i} = \dfrac{y-1}{y+1} = [-1, 1]$

$z = x + iy$

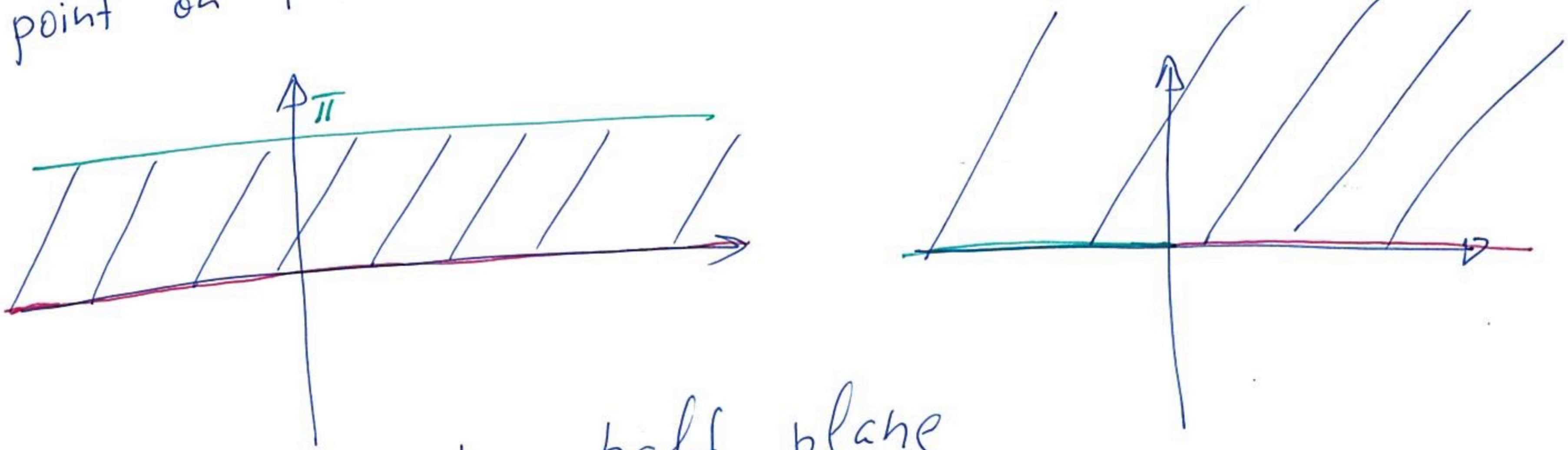
$$T(z) = \frac{x+iy-i}{x+iy-iy}$$

Supplementary to the example 10.50

## Another Example of conformal map strip to half plane

$f(z) = e^z$ $\qquad 0 < y < \pi$

point on the real axis are mapped to positive semi axis



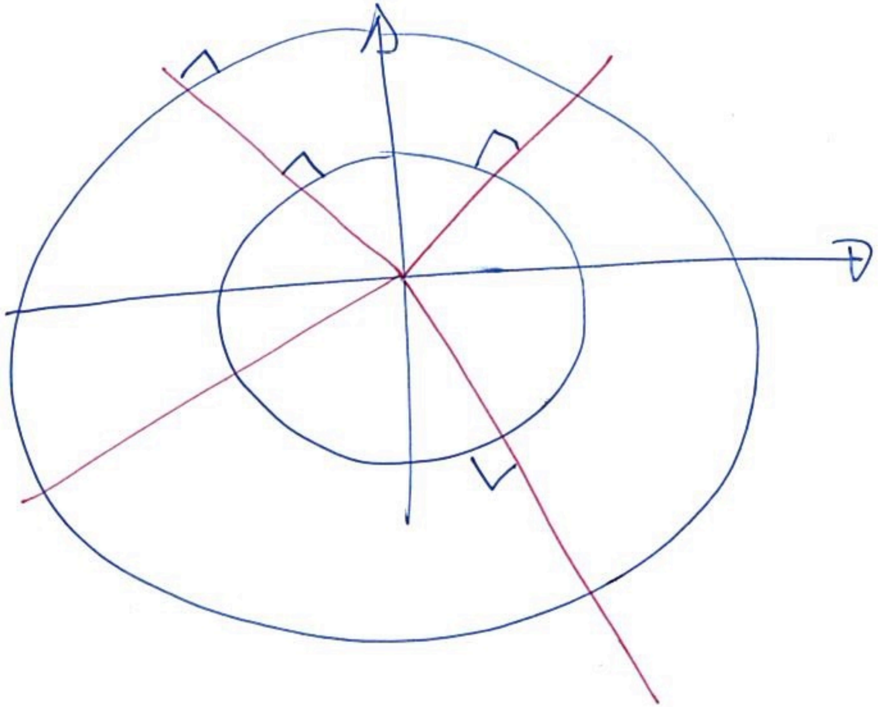

strip to half plane

Lines that form the grid

$z = x + ia \longrightarrow e^{x+ia} = e^{ia} e^{x}$ rays at angle $a$

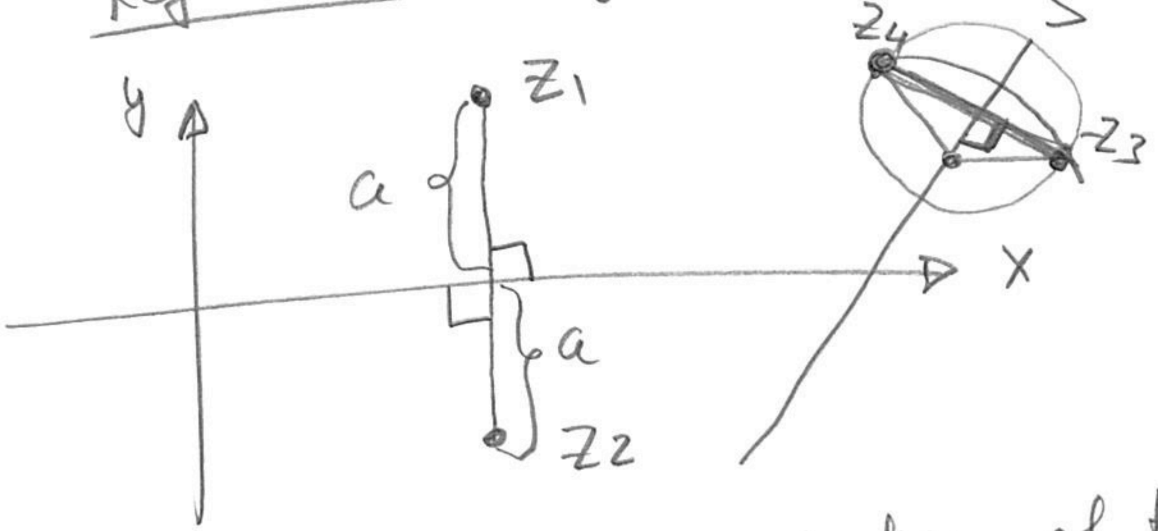
$z = b + iy \longrightarrow e^{b+iy} \rightarrow e^{b}(\cos y + i \sin y)$ circles of radius $e^b$

perpendicular



Next we consider Reflection and symmetry

Reflection and symmetry in a line




point $z_1$ is a reflection of the point $z_2$ in a line x. In other words it is symmetric with respect to x.

The point $z_3$ and $z_4$ are symmetric with respect to $S$.

We now discuss the reflection with respect to a circle.

Note that if two poins $z_3, z_4$ are symmetric with respect to a line $S'$ than any circe through $z_3$ and $z_4$ intersects $S'$ orthogonally.

Definition

Suppose $S$ is a line or a circle.
A pair of points $z_1$ and $z_2$ is called symmetric with respect to $S$ if every line or circle through the points $z_1$ and $z_2$ intersect $S$ orthogonally.

This definition mathes a symmetric points with respect to a straigh line where both properties hold.
But now it puts lines and circles on the same footing, and actually defines symmetry with respect to circles.

Note, that because ~~conformal transformatio~~ Möbius transformation ~~preserve angles~~ map lines and circles into lines and circles this is quite natural.
Morover, since they preserve angle between curves, point symmetric with respect to a line remain symmetric.
$T(z_1)$ and $T(z_2)$ are symmetric with respect to $T(S)$.

Theorem

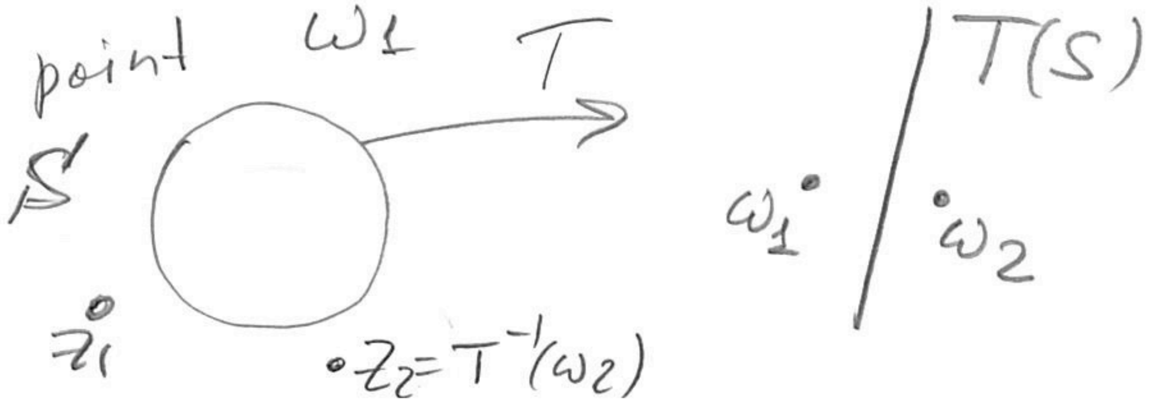
Suppose $S$ is a line or circle and
$z_1$ is a point not on $S$.
There is a unique point $z_2$ such that
the pair $z_1, z_2$ is symmetric in $S$.

Proof. Let $T$ be a fractional linear
transformation / Möbius transformation) that
maps $S$ to a line.
(if $S$ is a line to begin with this is
an identity transformation).

point $z_1 \xrightarrow{T(z_1)} \omega_1$

$\omega_1 = T(z_1)$

We know from elementary geometry
that there is one and only one reflection
with respect to a line $T(S)$ of the
point $\omega_1$



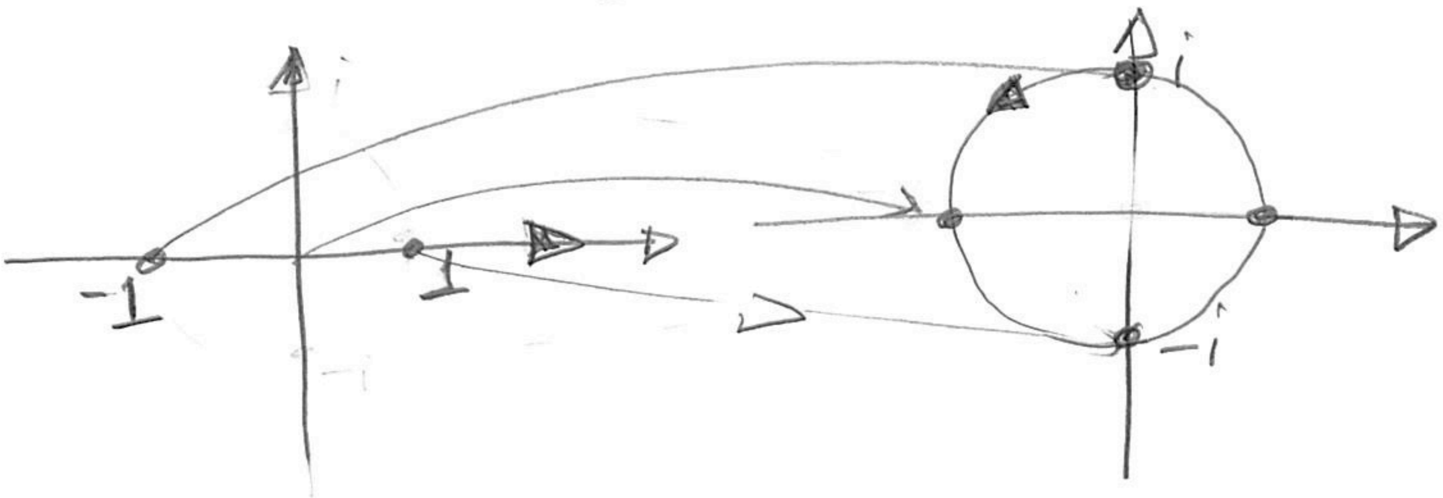
Since transformation $T$ is reversable we can define $z_2 = T^{-1}(\omega_2)$ as the reflection of $z_1$ in $S$.

Let us explore the geometric meaning of reflection in the circle.

Focus on the unit circle.

The transformation that maps the real axis onto the unit circle is

$$\frac{z-i}{z+i}$$



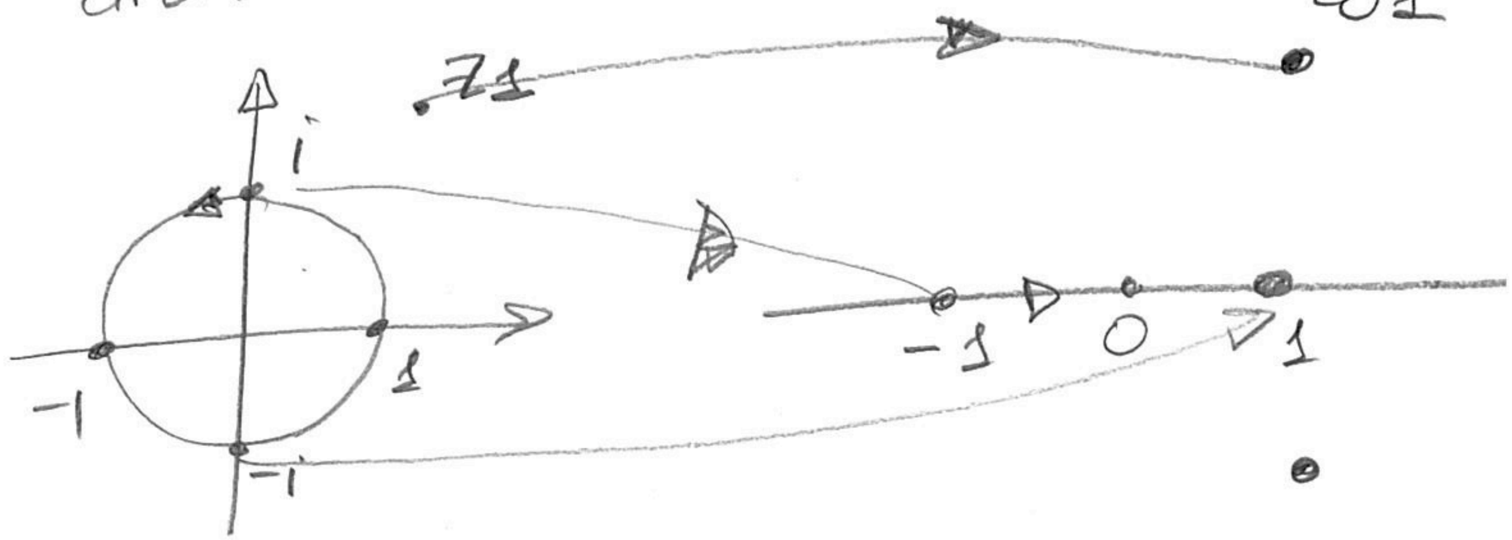
$$T(1) = \frac{1-i}{1+i} = \frac{(1-i)^2}{2} = \frac{\cancel{1}-\cancel{1}-2i}{2} = -i$$

$$T(-1) = \frac{-1-i}{-1+i} = + \frac{1+i}{1-i} = \frac{1}{-i} = i$$

$$T(0) = -1$$

Therefore we define a transformation

$$T(z) = -i\,\frac{z+1}{z-1}$$ that maps unit circle on to the real axis




$T(1) = \infty$

$$T(-1) = -i\,\frac{0}{-2} = 0$$

$$T(i) = -i\,\frac{1+i}{i-1} = \frac{1+i}{-i-1} = -1$$

$$T(-i) = -i\,\frac{1-i}{-1-i} = \frac{1-i}{-i+1} = 1$$

The point $z_1 \longrightarrow \omega_1 = -i\,\dfrac{z_1+1}{z_1-1}$

The reflection of $\omega_1$ with respect to the real axis is $\omega_2 = \overline{\omega_1} = i\,\dfrac{\overline{z_1}+1}{\overline{z_1}-1}$

Therefore the reflection of $z_1$ with respect to the unit circle

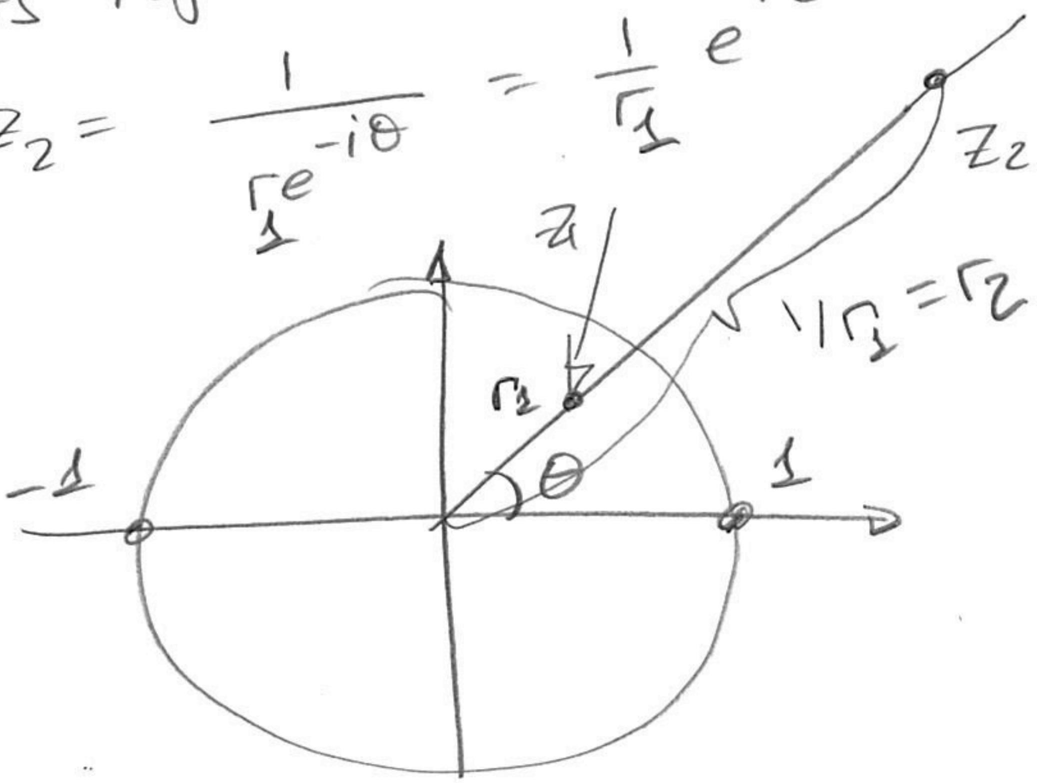
$$z_2 = T^{-1}(\omega_2) = \frac{\omega_2 - i}{\omega_2 + i} =$$

$$= \frac{i\,\dfrac{\overline{z_1}+1}{\overline{z_1}-1} - i}{i\,\dfrac{\overline{z_1}+1}{\overline{z_1}-1} + i} = \frac{(\overline{z_1}+1)-(\overline{z_1}-1)}{(\overline{z_1}+1)+(\overline{z_1}-1)} =$$

$$= \frac{2}{2\overline{z_1}} = \frac{1}{\overline{z_1}}$$

Therefore if the point $z_1 = r e^{i\theta}$ its reflection

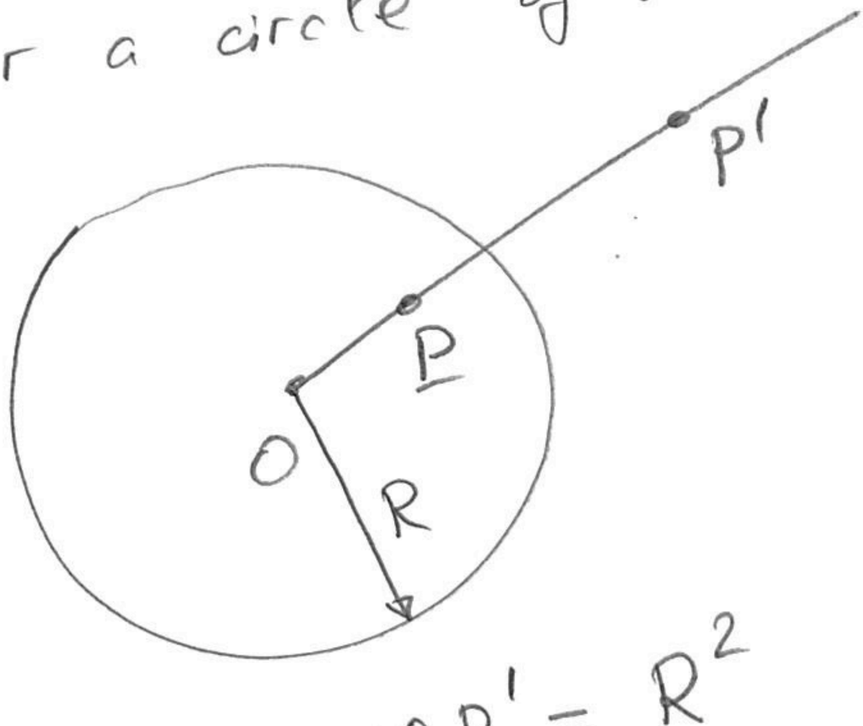
$$z_2 = \frac{1}{r_1 e^{-i\theta}} = \frac{1}{r_1} e^{i\theta}$$



The reflected point is on the radius vector at the distance $r_2 = \frac{1}{r_1}$ of the point $z_1$

In geometry it is called an inverse point. This construction emerges in inversive geometry.

For a circle of the radius $R$

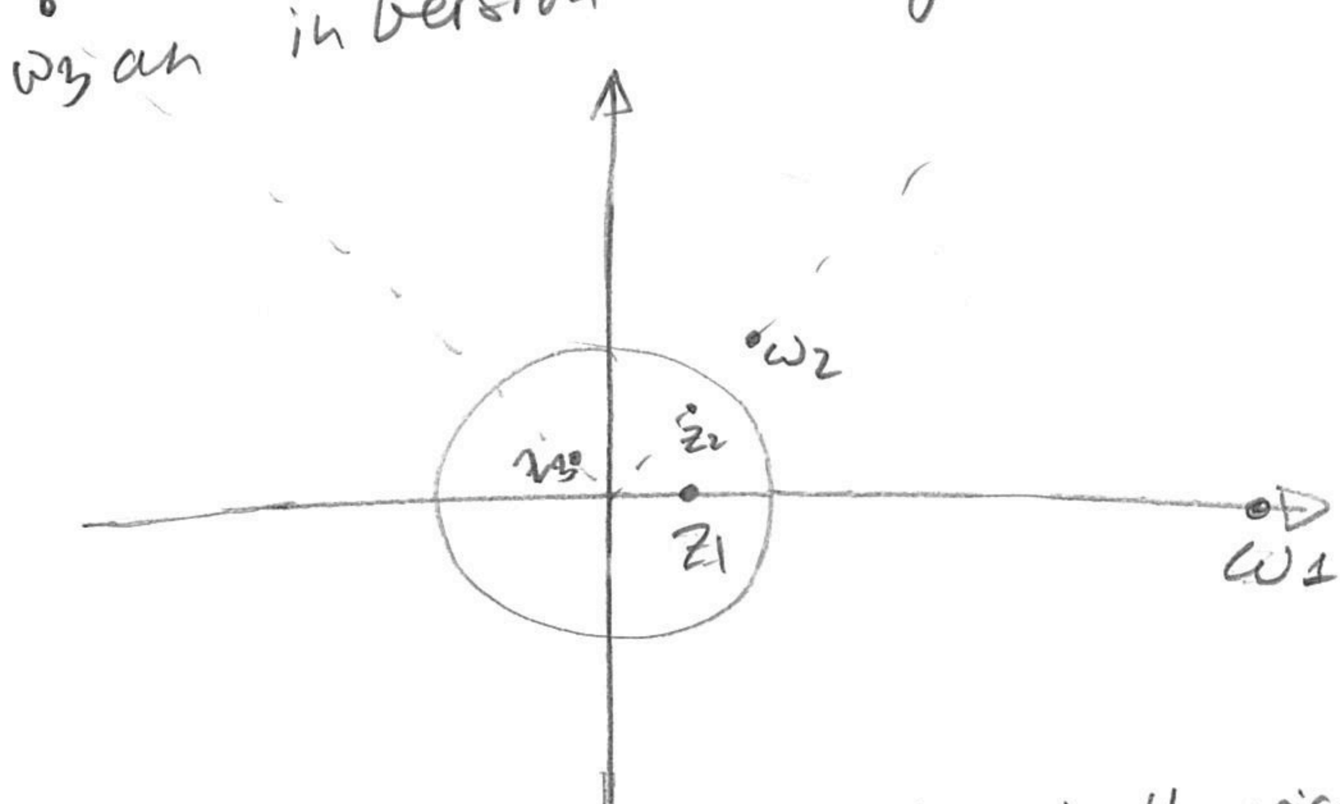



$$OP \cdot OP' = R^2$$

$$|OP'| = \frac{R^2}{|OP|}$$

Inversive geometry is the study of transformations of the Eucledian plane that maps circles or lines into circles or lines that preserve the angles between the crossing lines.

Note that we "discovered" inversive geometry from complex analysis "obvoiously", the same construction with a unit circle replaced by a circle of a finite radius agrees with an inversion with respect to a circle.




The closer one point is to the origin the further away is its reflection.
Origin maps to $\infty$.

# Dirichlet problem for harmonic functions

In general, a Dirichlet problem in a region D is to solve a diff. equation with the boundary conditions posed on the boundary of D, i.e. $\partial D$.

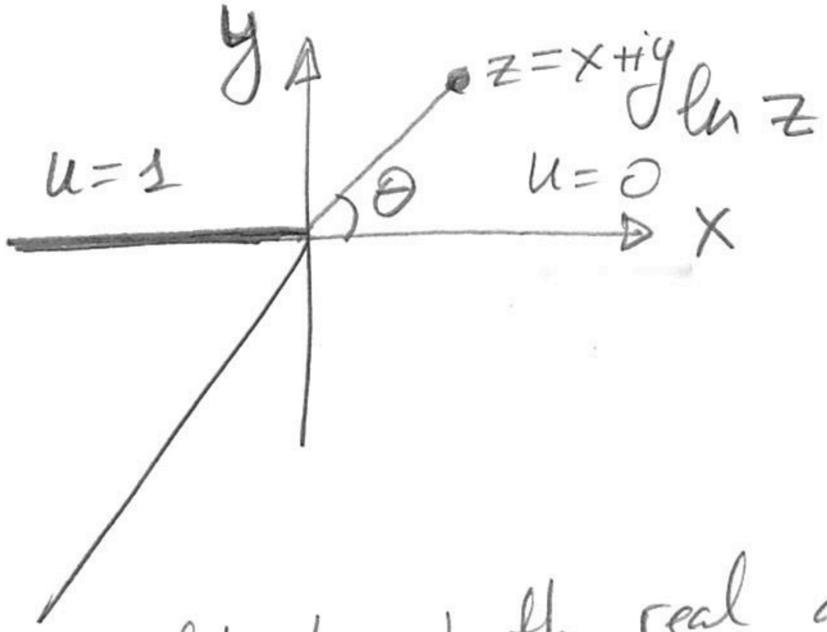
We will solve the Dirichel problem for the harmonic function, mapping the region D onto something simple, say upper half plane.

Example. Find a harmonic function $u(x,y)$, $\partial_x^2 u + \partial_y^2 u(x,y) = 0$ that satisfies the b.c. at $y=0$

$$u(x, y=0) = \theta(x)$$

Solution.

Function $\log z = \log|z| + i \arg z$ is analytic in the upper half plane [choose the cut into a lower half plane]




$z=x+iy$

By analiticity, both real and Imag. parts are harmonic function.

Therefore

$$\mathrm{Re}\left(\frac{1}{i\pi}\ln z\right) = \frac{\arg z}{\pi}$$

if we use standard notations

$\arg(x>0) = 0$

$\arg(x<0) = \pi$

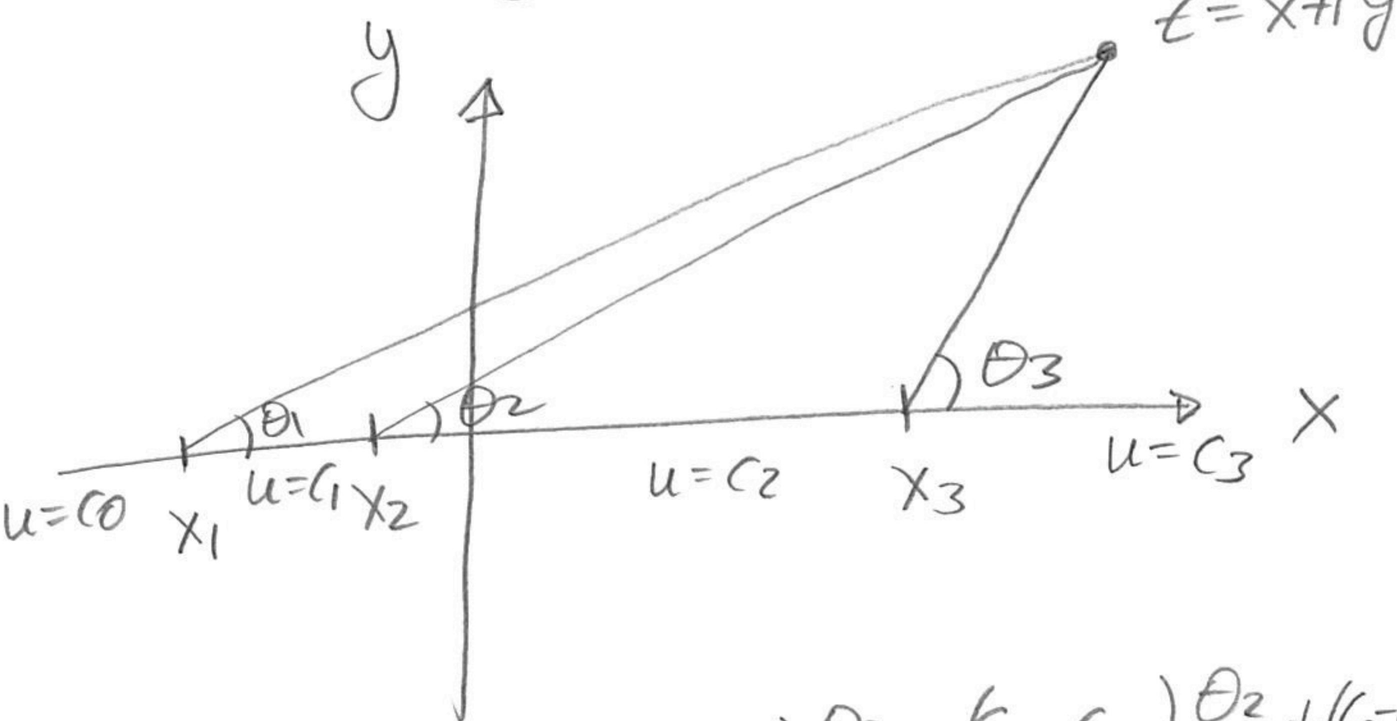
$$u(x,y) = \mathrm{Re}\left(\frac{1}{i\pi}\ln z\right)$$

This function has the righ b.c. and is harmonic.

## Example

Suppose $x_1 < x_2 < x_3$.
Find a harmonic function $u(x,y)$ that satisfies the boundary conditions

$$u(x, y=0) = \begin{cases} c_0, & x < x_1 \\ c_1, & x_1 < x < x_2 \\ c_2, & x_2 < x < x_3 \\ c_3, & x_3 < x \end{cases}$$




$$u(x,y) = c_3 + (c_2 - c_3)\frac{\theta_3}{\pi} + (c_1 - c_2)\frac{\theta_2}{\pi} + (c_0 - c_1)\frac{\theta_1}{\pi}$$

Using the same branch as before

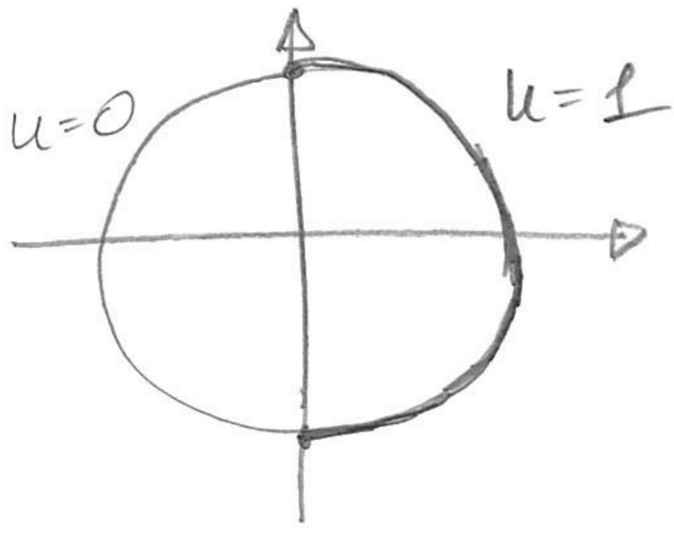
$$u(x,y) = \mathrm{Re}\left\{ c_3 + \frac{(c_2-c_3)}{i\pi} \ln(z-x_3) + \right.$$

$$\left. + \frac{(c_1-c_2)}{i\pi} \ln(z-x_2) + \frac{c_0-c_1}{i\pi} \ln(z-x_1) \right\}$$

## Example

Find a harmonic function $u(x,y)$ within unit circle, such that on the unit circle $x^2+y^2=1$

$$u(z) = \begin{cases} 1, & \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$



Let us use a map that send a unit circle onto upper half plane

$$\omega = -i\,\frac{z+1}{z-1} = T(z)$$

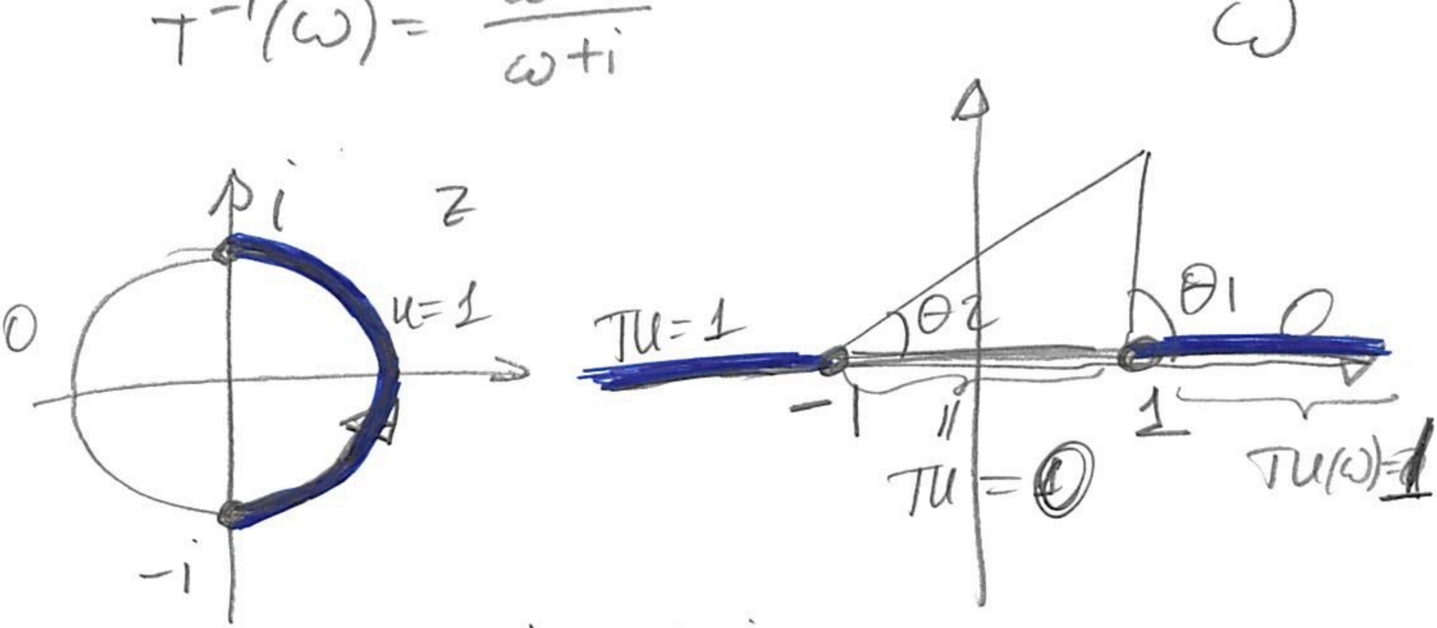
The inverse transformation

$$(z-1)\,\omega + i\,(z+1) = 0$$

$$z\,(\omega+i) + (i-\omega) = 0$$

$$z = -\,\frac{i-\omega}{i+\omega} = T^{-1}(\omega)$$

$$T^{-1}(\omega) = \frac{\omega - i}{\omega + i}$$




$$T(i) = -i\,\frac{1+i}{i-1} = \frac{1+i}{-1-i} = -1$$

$$T(-i) = -i\,\frac{1-i}{-1-i} = \frac{1-i}{-i+1} = 1$$

$$T(-1) = 0$$

$\theta_1 = \theta_2 = 0 \longrightarrow 1$

$\theta_1 = \theta_2 = \pi \longrightarrow 1$

$\theta_1 = \pi \quad \theta_2 = 0 \longrightarrow 0$

$$U = -\frac{\theta_1}{\pi} + \frac{\theta_2}{\pi} + 1 =$$

$$= \mathrm{Re}\left\{ 1 + \frac{\ln \overbrace{(w+1)}^{\theta_2}}{\pi i} - \frac{\ln \overbrace{(w-1)}^{\theta_1}}{\pi i} \right\}$$

On the $w$ plane this is the solution.

Map it back onto $z$ plane

$$U(z) = \mathrm{Re}\left\{ 1 + \frac{1}{\pi i} \ln\left(-i\frac{z+1}{z-1} + 1\right) - \frac{1}{\pi i} \ln\left(-i\frac{z+1}{z-1} - 1\right) \right\} =$$

$$= 1 + \mathrm{Re}\left\{ \frac{1}{\pi i} \ln \frac{-iz-i+z-1}{z-1} - \frac{1}{\pi i} \ln \frac{-iz-i-z+1}{z-1} \right\}$$