Fourier transform

Lecture 11

$f(x)$

Define

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}$$

Note that

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{+ikx}\, \tilde{f}(k)$$

check

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} \int_{-\infty}^{\infty} dx_1\, e^{-ikx_1} f(x_1) =$$

$$= \int_{-\infty}^{\infty} dx_1 \underbrace{\int \frac{dk}{2\pi}\, e^{ik(x-x_1)}}_{\delta(x-x_1)} f(x_1) =$$

$$= \int_{-\infty}^{\infty} dx_1\, \delta(x-x_1)\, f(x_1) = f(x)$$

More carefully

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,e^{ik(x-x_1)} = \int_{-\infty}^{0}\frac{dk}{2\pi}\,e^{k(i(x-x_1)+\delta)} +$$

$$+\int_{0}^{\infty}\frac{dk}{2\pi}\,e^{k(i(x-x_1)-\delta)} =$$

$$= \frac{1}{2\pi}\,\frac{e^{k(i(x-x_1)+\delta)}}{i(x-x_1)+\delta}\Bigg|_{-\infty}^{0} + \frac{1}{2\pi}\,\frac{e^{k(i(x-x_1)-\delta)}}{i(x-x_1)-\delta}\Bigg|_{0}^{\infty}$$

$$= \frac{1}{2\pi}\,\frac{+1}{i(x-x_1)+\delta} - \frac{1}{2\pi}\,\frac{1}{i(x-x_1)-\delta} =$$

$$= \frac{1}{2\pi i}\,\frac{1}{x-x_1-i\delta} - \frac{1}{2\pi i}\,\frac{1}{x-x_1+i\delta} =$$

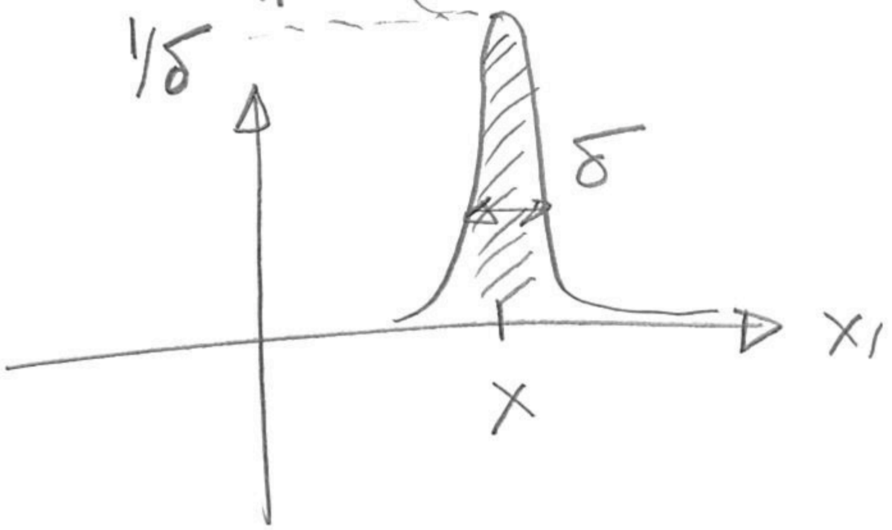
$$= \frac{1}{2\pi i}\left[\frac{1}{x-x_1-i\delta} - \frac{1}{x-x_1+i\delta}\right]$$

$\delta \to 0$

$$\delta(x-x_1) = \frac{1}{2\pi i}\left[\frac{1}{x-x_1-i\delta} - \frac{1}{x-x_1+i\delta}\right] =$$

$$= \frac{2i\delta}{(2\pi i)(x-x_1-i\delta)(x-x_1+i\delta)} =$$

$$= \frac{1}{\pi}\frac{\delta}{(x-x_1)^2+\delta^2}$$



area under the function = 1

$$\int dx_1 f(x_1)\,\frac{1}{\pi}\frac{\delta}{(x-x_1)^2+\delta^2} \overset{\delta\to 0}{=} f(x)$$

In a more general form the function

$$\frac{1}{x-x_1+i0} = -\pi i\,\delta(x-x_1) + \frac{P}{x-x_1}$$

$$\frac{1}{x-x_1-i0} = \text{c.c. of} = \pi i\,\delta(x-x_1) + \frac{P}{x-x_1}$$

Sohotski-Plemelj theorem

$$\frac{1}{x-x_1-i0} - \frac{1}{x-x_1+i0} = 2\pi i\,\delta(x-x_1)$$

Dirac delta function

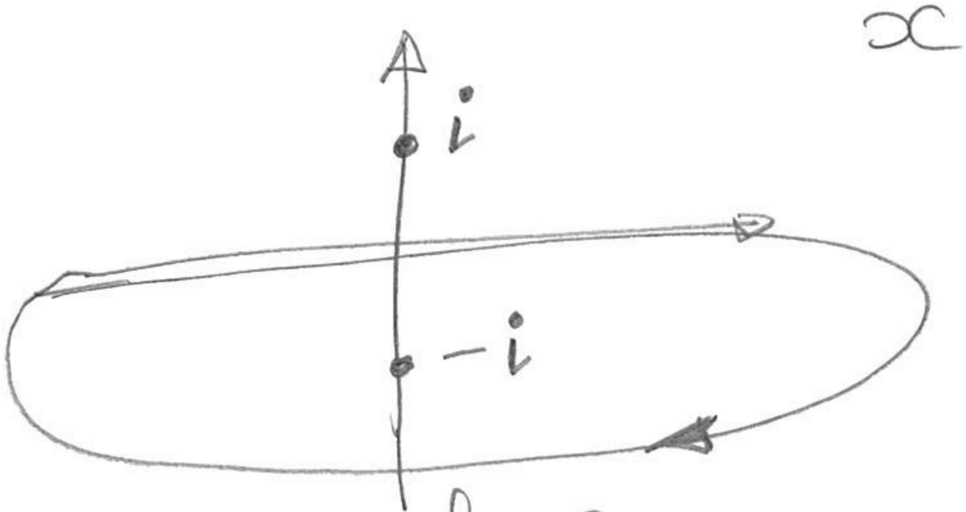
Next we compute FT for simple cases

Example

$$f(x) = \frac{1}{x^2+1} = \frac{1}{(x+i)(x-i)}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx} =$$

$$= \int_{-\infty}^{\infty} dx\, \frac{e^{-ikx}}{(x+i)(x-i)}$$



Res in $x=-i$

Assume $k>0$.

Close the integral in the lower half plane, because it converges there

$$\tilde{f}(k) = -2\pi i\, \frac{e^{-i(-i)k}}{(-2i)} =$$

$$= \frac{-2\pi i}{-2i} e^{-k} = \pi e^{-k} \quad \text{for } k>0$$

Next study $k<0$
Close the integral in the upper half plane

$$\tilde{f}(k) = 2\pi i \, \frac{e^{-iki}}{2i} = \pi e^{k}$$

Therefore, for all k's

$$\tilde{f}(k) = \pi e^{-|k|}$$

Example

$$f(x) = e^{-x^2/2a}, \quad a>0$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx \, e^{-ikx - x^2/2a} =$$

$$= \int_{-\infty}^{\infty} dx \, e^{-\left(\frac{x}{\sqrt{2a}} + i\frac{\sqrt{2a}}{2}k\right)^2 - \frac{2a}{4}k^2} =$$

$$= e^{-\frac{ak^2}{2}} \cdot \int_{-\infty}^{\infty} dx \, e^{-\left(\frac{x}{\sqrt{2a}} + i\sqrt{\frac{a}{2}}\,k\right)^2} =$$

$$\frac{x}{\sqrt{2a}} = \tilde{x}$$

$$dx = d\tilde{x} \cdot \sqrt{2a}$$

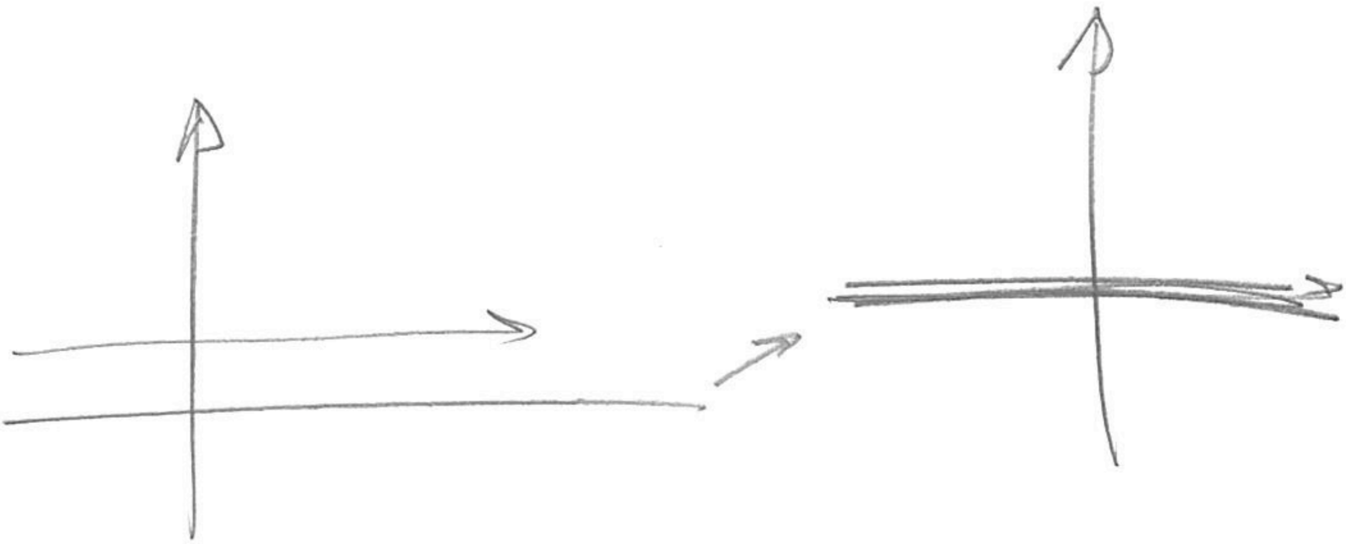
$$\tilde{f}(k) = e^{-\frac{ak^2}{2}} \sqrt{2a} \int_{-\infty}^{\infty} d\tilde{x}\, e^{-\left(\tilde{x} + i\sqrt{\frac{a}{2}}\, k\right)^2} =$$

$$\tilde{x} \to \tilde{\tilde{x}} - i\sqrt{\frac{a}{2}}\, k$$

$$= e^{-\frac{ak^2}{2}} \sqrt{2a} \int_{-\infty - i\sqrt{\frac{a}{2}}\, k}^{\infty - i\sqrt{\frac{a}{2}}} d\tilde{\tilde{x}}\, e^{-\tilde{\tilde{x}}^2}$$

But I can <u>move</u> the countur

$$\int_{-\infty - i\sqrt{\frac{a}{2}}\, k}^{\infty - i\sqrt{\frac{a}{2}}} d\tilde{\tilde{x}}\, e^{-\tilde{\tilde{x}}^2} = \int_{-\infty}^{\infty} dx\, e^{-x^2}$$



Trick to compute

$$I = \int_{-\infty}^{\infty} dx\, e^{-x^2} \text{ is to compute}$$

$$I^2 = \left(\int dx\, e^{-x^2}\right)\left(\int dy\, e^{-y^2}\right) =$$

$$= \int dx\,dy\, e^{-(x^2+y^2)} = \int_0^{2\pi} d\varphi \int_0^{\infty} r\,dr\, e^{-r^2} =$$

$$= 2\pi \int_0^{\infty} r\,dr\, e^{-r^2} = 2\pi \int_0^{\infty} d\left(r^2/2\right) e^{-r^2} =$$

$$= \frac{2\pi}{2} \int_0^{\infty} dz\, e^{-z} = \pi\left(-e^{-z}\right)\Big|_0^{\infty} = \pi$$

$$I^2 = \pi \longrightarrow I = \sqrt{\pi}$$

$$\tilde{f}(k) = \sqrt{2\pi a}\; e^{-ak^2/2}$$

How FT helps?

Assume we have a differential eq. in which objects like $\frac{df(x)}{dx}$, $\frac{d^2 f(x)}{dx^2}$ etc. appear.

Let us compute the FT of such an object, i.e. $FT\left(\frac{df}{dx}\right)$

$$\int_{-\infty}^{\infty} dx \underbrace{\left(\frac{d}{dx} f(x)\right)}_{df(x)} \underbrace{e^{-ikx}}_{e^{-ikx}} = f(x) e^{-ikx} \Big|_{-\infty}^{\infty} -$$

$$- \int_{-\infty}^{\infty} f(x) \, d e^{-ikx} = \underbrace{f(x) e^{-ikx} \Big|_{-\infty}^{\infty}}_{\substack{\text{if } f(\pm\infty)=0 \\ \text{drops}}} + \underbrace{ik \int f(x) e^{-ikx} dx}_{FT(f(x))} =$$

$$= ik \tilde{f}(k)$$

Similarly

$$FT\left(\frac{d^2}{dx^2} f(x)\right) = (ik)^2 \tilde{f}(k)$$

and in general $FT\left(\frac{d^n}{dx^n} f\right) = (ik)^n \tilde{f}(k)$

Example

First order ODE + bound. condition $f(x)=0$ for $x<0$

$$\frac{df(x)}{dx} = \Theta(x)\, e^{-\delta x}$$

$$ik\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, \Theta(x)\, e^{-ikx-\delta x} = \int_0^{\infty} dx\, e^{-ikx-i}=$$

$$= \int_0^{\infty} dx\, e^{x(-ik-\delta)} = \left.\frac{e^{-x(+ik+\delta)}}{-(ik+\delta)}\right|_0^{\infty}$$

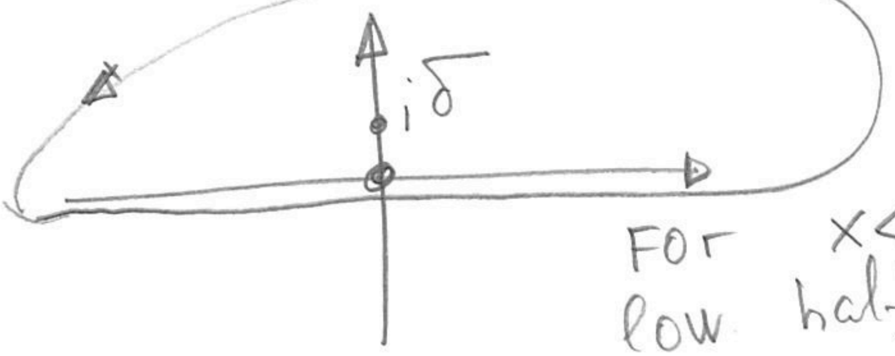
$$= \frac{1}{ik+\delta}$$

$$\tilde{f}(k) = \frac{1}{(ik)(ik+\delta)} = \frac{+1}{k(-k+i\delta)} =$$

$$= \frac{-1}{k(k-i\delta)}$$

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, \frac{-1}{k(k-i\delta)} \rightarrow$$

$$\rightarrow -\int \frac{dk}{2\pi} \frac{e^{ikx}}{(k-i0)(k-i\delta)}$$



For $x<0$ close in the low half plane and get zero.

For $x > 0$

$$f(x) = \cancel{2\pi} i \frac{1}{\cancel{2\pi}}(-1)\left[\underbrace{\frac{1}{-i\delta}}_{k=0} + \underbrace{\frac{e^{-\delta x}}{i\delta}}_{k=i\delta}\right] =$$

$$= \frac{-i}{i\delta}\left[-1 + e^{-\delta x}\right] = -\frac{1}{\delta}\left(-1 + e^{-\delta x}\right) =$$

$$= \frac{1 - e^{-\delta x}}{\delta}$$

Of course much simpler calculation

$$f(x) = \int_{-\infty}^{x} dx_1 \, \theta(x_1) e^{-\delta x_1} = \quad \text{assuming } x > 0$$

$$= \int_{0}^{x} dx \, e^{-\delta x} = \left.\frac{e^{-\delta x}}{-\delta}\right|_{0}^{x} = \frac{e^{-\delta x} - 1}{-\delta} =$$

$$= \frac{1 - e^{-\delta x}}{\delta}.$$ We got the same.

However we can now use FT in the same way for high order ODE.