Complex analys Lecture 2

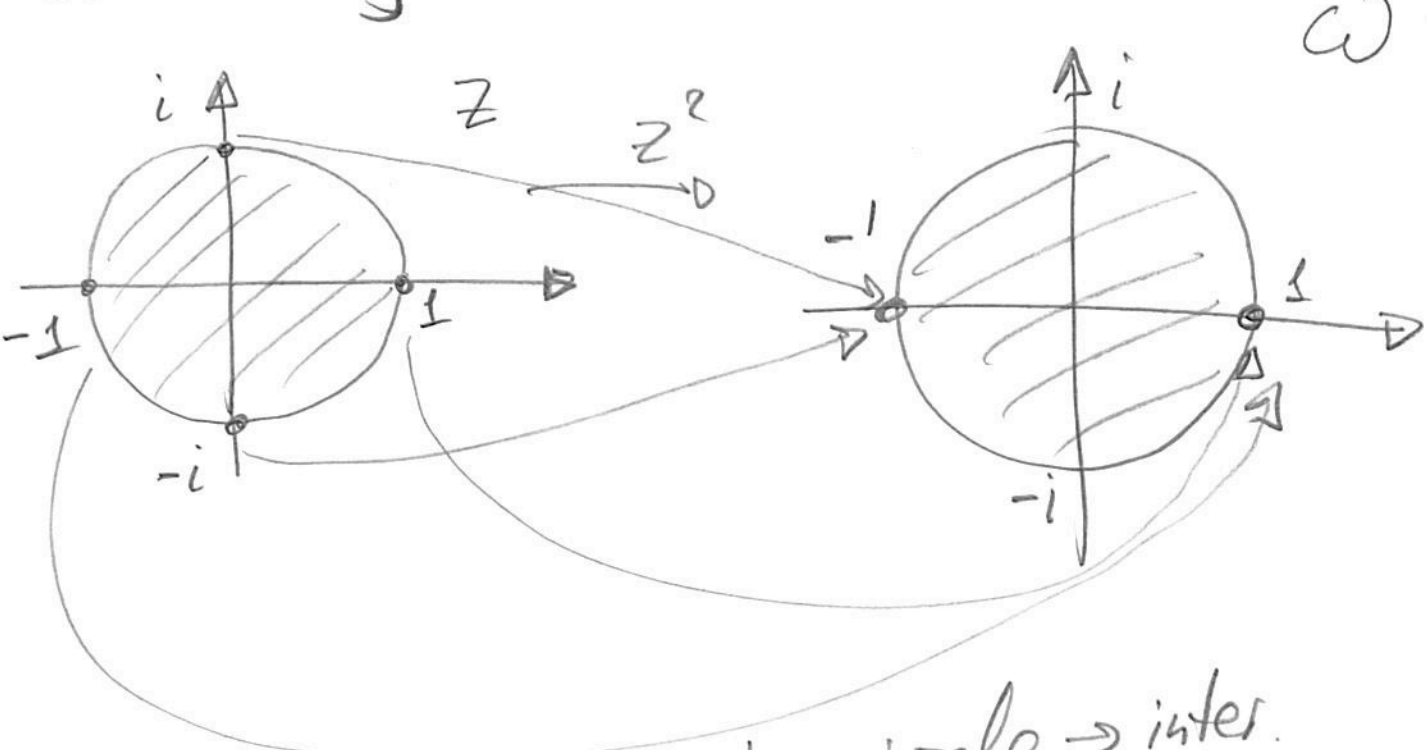
$f(z) = u(x,y) + i v(x,y)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

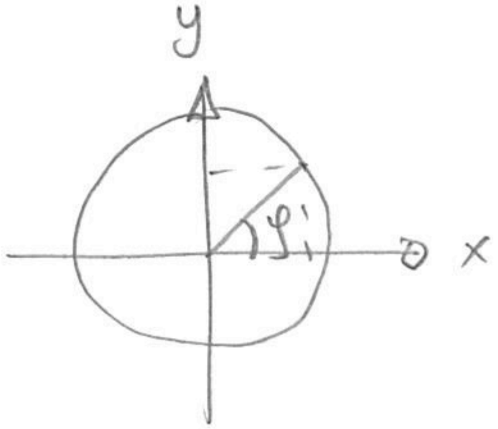
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Function $\omega(z) = z^n$

Consider $f(z) = z^2 = \omega$ and $\omega(z) = \sqrt{z}$



interiour of the unit circle → inter.
exterior of the unit circle → exter
the point that goes once rounde
the circle → point that goes
twice



$z = x + iy = \cos\varphi + i\sin\varphi = e^{i\varphi}$

$\varphi \in$ Real $\quad z \in$ Unit circle

$\cos\varphi = \dfrac{e^{i\varphi} + e^{-i\varphi}}{2} \qquad \sin\varphi = \dfrac{e^{i\varphi} - e^{-i\varphi}}{2}$

consistent with the definition of $\cos\varphi$, $\sin\varphi$ for real function if to understand

$e^x = 1 + x + \frac{x^2}{2} + \dots \frac{x^n}{n!}$

For complex number $z = x + iy$

$\omega = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

Obciously that for $y = 0$, i.e. $z \in$ Real

$e^z = e^x$ and the definion reduces to the familiar one

Also

$$e^{z_1} e^{z_2} = e^{x_1+iy_1} e^{x_2+iy_2} =$$

$$= e^{x_1}(\cos y_1 + i\sin y_1)\, e^{x_2}(\cos y_2 + i\sin y_2) =$$

$$= e^{x_1+x_2}(\cos y_1 \cos y_2 + i\cos y_1 \sin y_2 + i\sin y_1 \cos y_2$$

$$- \sin y_1 \sin y_2) =$$

$$= e^{x_1+x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2)) =$$

$$= e^{x_1+x_2}\, e^{i(y_1+y_2)} = e^{x_1+x_2+i(y_1+y_2)} = e^{z_1+z_2}$$

Derivative

$$e^{x+iy} = e^x(\cos y + i\sin y) = \underbrace{e^x \cos y}_{u} + i\underbrace{e^x \sin y}_{v}$$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ Cauchi-Rieman cond. hold

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y =$$

$$= e^x(\cos y + i\sin y) = e^z$$

$$\boxed{\frac{de^z}{dz} = e^z}$$

The representation

$$z = re^{i\varphi} = \underbrace{r\cos\varphi}_{x} + i\underbrace{r\sin\varphi}_{y}$$

Allows to represent any number on the complex plane by the module r and the argument φ $\quad (r, \varphi)$

Note that $\quad \varphi \to \varphi + 2\pi$

$$z \to z$$

Therefore if

$$\boxed{e^{z_1} = e^{z_2} \to z_1 - z_2 = 2\pi i k \quad k \in \text{Integer}}$$

The inverse function to the exponent is logarithm

$$e^{\omega} = z \longrightarrow \omega = \ln z$$

multivalue function

$$e^{\omega_1} = z_1 \qquad e^{\omega_2} = z_2$$

$$e^{\omega_1 + \omega_2} = z_1 z_2$$

$$\ln(z_1 z_2) = \omega_1 + \omega_2 = \ln z_1 + \ln z_2$$

$z_1 = |z|$ $\quad \varphi = \arg z$

$z_2 = e^{i\varphi} = e^{i \arg z}$

$\ln(|z| e^{i\arg z}) = \ln|z| + \ln e^{i \arg z} =$

$= \ln|z| + i \underbrace{(\arg z + 2\pi k)}_{\text{Arg } z}$

an infinite set of points

$\ln z = \ln|z| + i \,\text{Arg}\, z$

ln is a multivalued function

note that $\sqrt{z}$ is also multi-valued function

$z = re^{i\varphi}$

$\sqrt{z} = \sqrt{r}\, e^{i\varphi/2}$ and also

$\sqrt{z} = \sqrt{r}\, e^{\frac{i(\varphi+2\pi)}{2}} = \sqrt{r}\, e^{i\varphi/2 + i\pi} = -\sqrt{r}\, e^{\frac{i\varphi}{2}}$

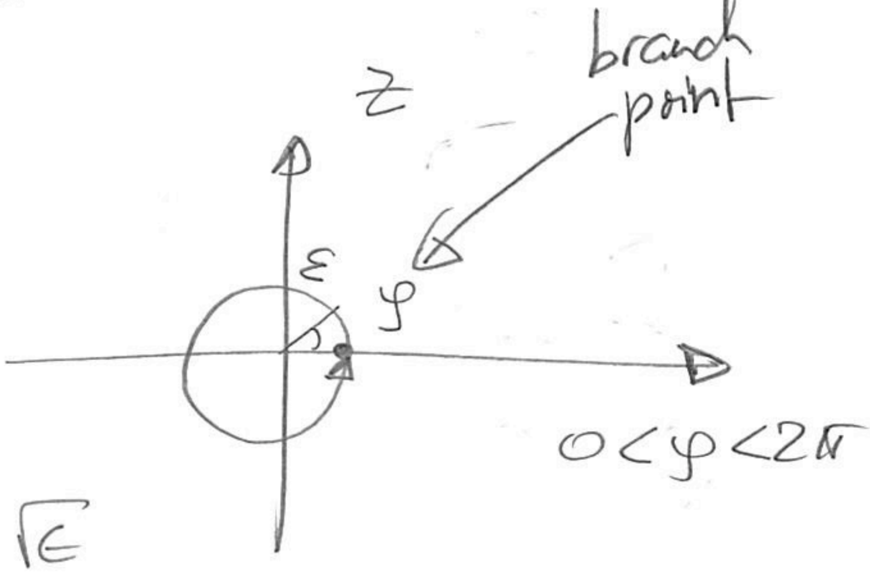
$\sqrt{z}$ is two valued function

$\ln z$ is ∞ valued function

Can we look at a single brangh of these functions?

Focus on $z^{1/2}$ and decide that

$\sqrt{z} = \sqrt{r}\, e^{i\varphi/2}$



$\varphi = 0 \qquad \sqrt{z} = \sqrt{\epsilon}$

$\sqrt{z} = \sqrt{\epsilon e^{i\varphi}} = \sqrt{\epsilon}\; e^{i\varphi/2} \xrightarrow{\varphi \to 2\pi}$

here we assumed the fixed branch

$\to \sqrt{\epsilon}\, e^{i\frac{2\pi}{2}} = \sqrt{\epsilon}\, e^{i\pi} = -\sqrt{\epsilon}$

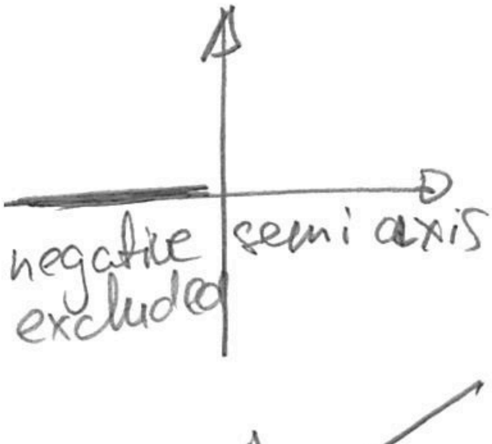
So we have continously moved by an infinitezimal ammound that

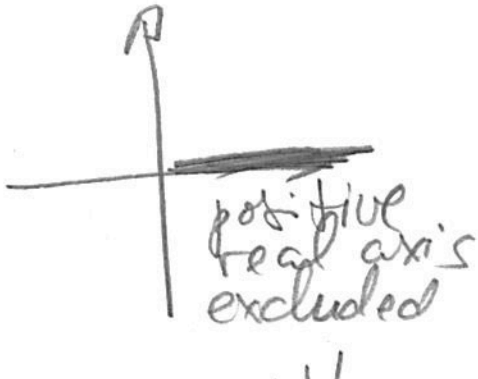
$\sqrt{z} = -\sqrt{r}\, e^{i\varphi/2}$, that is the second branch. This means that we can not separate the two branches in the domain that includes the origin

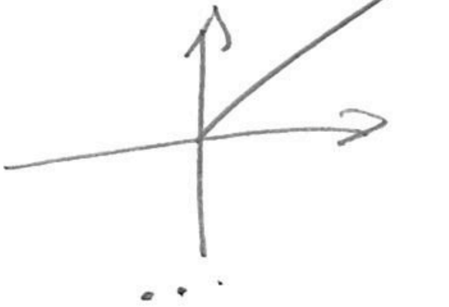
On the other hand

in this domain we can choose a branch of $\sqrt{z}$ and it will never turn into the second branch on any closed trajectory C

Actualy, we need to exclude the possibility of encompassing the origin, so the following domains are also ok



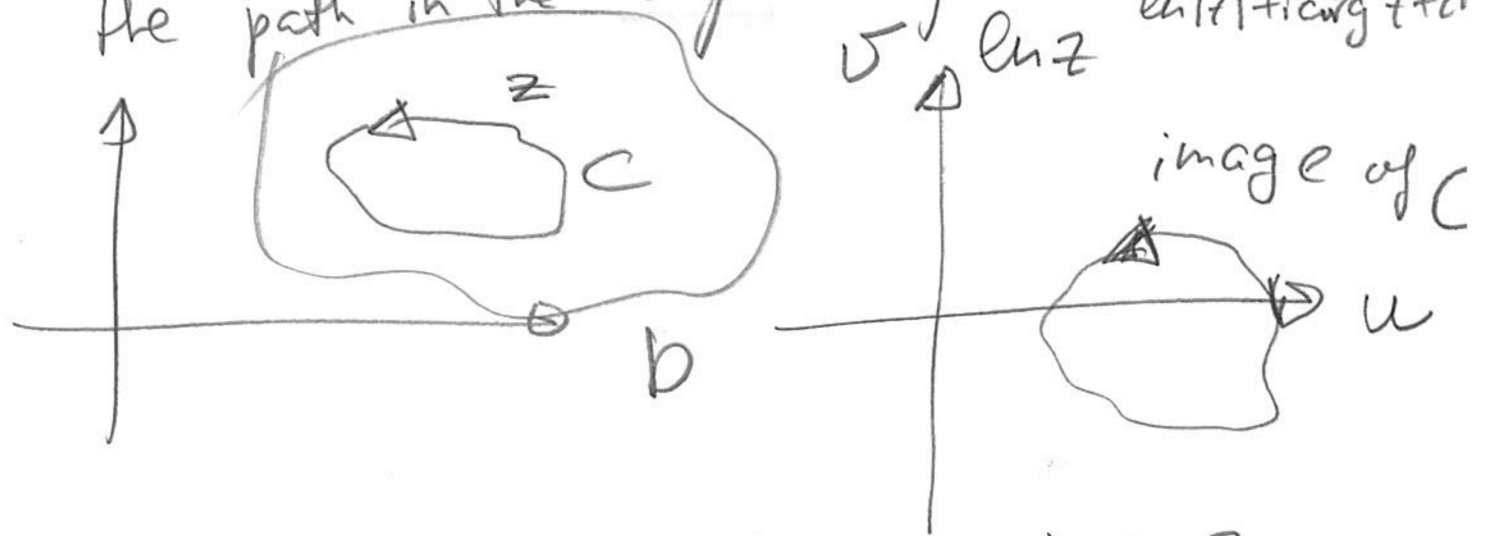




complex plane with a cut

...

Conside the l[redacted]z cend consider D the path in the complex plane



$$\ln z = u + iv = \ln|z| + i\arg z$$

the $0 < \varphi < 2\pi$

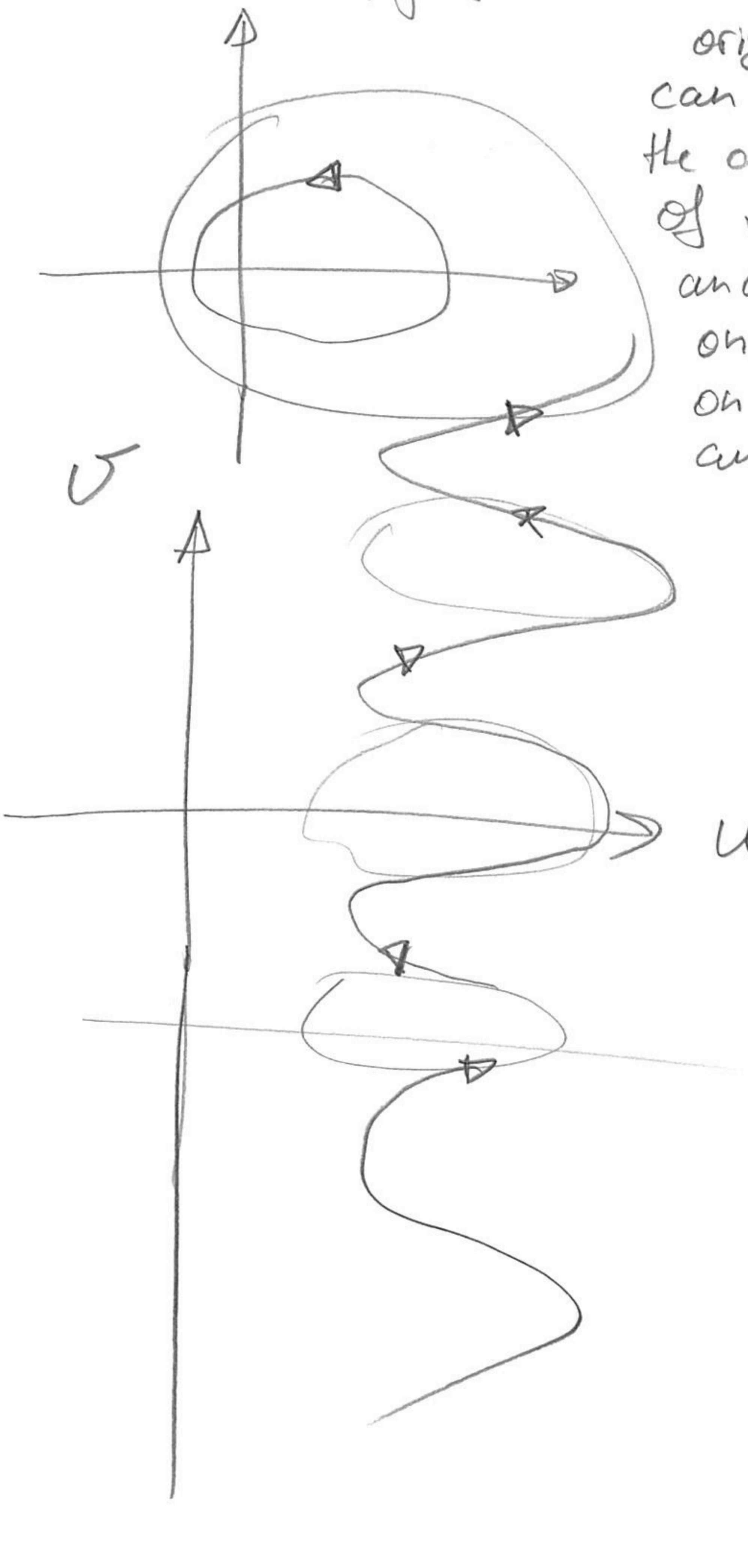
$\ln|z| + i\arg z - 2\pi i$

all the branche perform a "circle" ~~separate~~ shifted by $2\pi i k$. Each branch is a single valued function in D

$$\frac{d \ln z}{dz} = \frac{1}{(e^{\omega})'} = \frac{e^{\omega}}{e^{2\omega}} = \frac{1}{e^{\omega}} = \frac{1}{z}$$ and the branches of log are continous, differ. functions

here we used $y'_x = 1/x_y$

If the domain containt the origin than we can not separet the different bran of the logarithm and aroud $z=0$ one goes from one branch onto another.



A general function

$$\omega = z^{\alpha} = e^{\alpha \ln z}$$ has the

sape structure as log if α is not rational.

if $\alpha =$ integer, say 2

$$\omega = z^2 = e^{2\ln z} = e^{2(\ln|z| + i\varphi + 2\pi i k)} =$$

$$= e^{2\ln|z| + 2i\varphi + 4\pi i k} = e^{2\ln z + 2i\varphi}$$

and it is a single valued function as it should.

if $\alpha =$ rattional $\alpha = p/q$, (say $\alpha = \frac{1}{2}$)

$$\omega = z^{1/2} = e^{\frac{1}{2}\ln z} = e^{\frac{1}{2}(\ln|z| + i\varphi + 2\pi i k)} =$$

$$= e^{\frac{1}{2}\ln|z| + \frac{i\varphi}{2} + \pi i k} = \begin{cases} e^{\frac{1}{2}\ln|z| + \frac{i\varphi}{2}} & k \in \text{even} \\ -e^{\frac{1}{2}\ln|z| + \frac{i\varphi}{2}} & k \in \text{odd} \end{cases}$$

all

Again if D excuded origin $\log z$ and z^{α} are single valued

$$z^{p/q} = e^{\frac{p}{q}(\ln|z| + i\varphi + 2\pi i k)} =$$

$$= e^{\frac{p}{q}\ln|z| + i\varphi\frac{p}{q} + 2\pi i k \cdot p/q}$$

In this ~~case~~ case there are q branches

$0 \le k < q-1$

(**) Definitions: To separate the branches one introduces cuts — lines that are forbidden to cross. After the introduction of the branch cuts, each branch of a multi-valued ~~analytic~~ function defines a single valued function that is analytic everywhere except at the branch cut, where it is discontinuous.

(**) A function $f(z)$ is analytic in the domain D of a complex plain if $f(z)$ has a derivative at each point in D, and is single valued