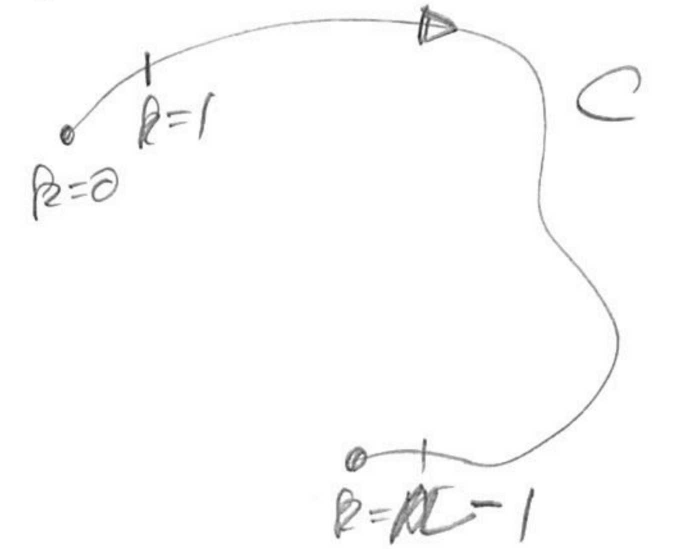
Integral of the complex function

Lecture 3

$f(z)$



oriented curve

$$\int_C f(z)dz = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(z_k)(z_{k+1}-z_k)$$

$f(z) = u(x,y) + i\,v(x,y)$

$z_k = x_k + i y_k$

$x_{k+1} - x_k = \Delta x_k$

$y_{k+1} - y_k = \Delta y_k$

$$\int_C f(z)dz = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(u(x_k,y_k) + i\,v(x_k,y_k) \right) \cdot (\Delta x_k + i\Delta y_k) =$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} \Big(\big[u(x_k,y_k)\,\Delta x_k - v(x_k,y_k)\,\Delta y_k\big] + i\big[u(x_k,y_k)\,\Delta y_k + v(x_k,y_k)\,\Delta x_k\big] \Big) =$$

$$= \int_C \big[u(x,y)\,dx - v(x,y)\,dy\big] + i\big[u(x,y)\,dy + v(x,y)\,dx\big]$$

Properties of the integral of the complex function

If $t \in$ Real

$\omega(t) = \varphi(t) + i\psi(t)$

$\frac{d\omega}{dt} = \varphi'(t) + i\psi'$

$$\int_a^b dt\,\omega(t) = \int_a^b dt\,\varphi(t) + i\int_a^b dt\,\psi(t)$$

If we parametrize C by t

$$\int_C dz\, f(z) = \int_a^b dt\, \frac{dz}{dt} f(z(t))$$

Linearity

$$\int_C dz\, [a f(z) + b g(z)] = a \int_C dz\, f(z) + b \int_C dz\, g(z)$$

$$\int_{C_1+C_2} dz\, f(z) = \int_{C_1} dz\, f(z) + \int_{C_2} f(z)\, dz$$

$$\int_C dz\, f(z) = -\int_{C^-} dz\, f(z)$$

C^- the same curve with an opposite orientation

Cauchy's integral theorem

Let D be a simply connected open domain.

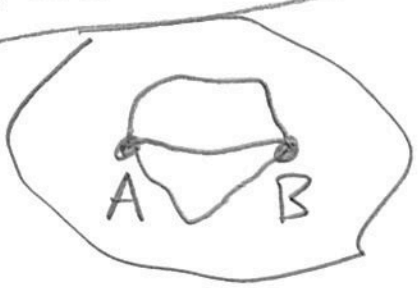
~~set~~

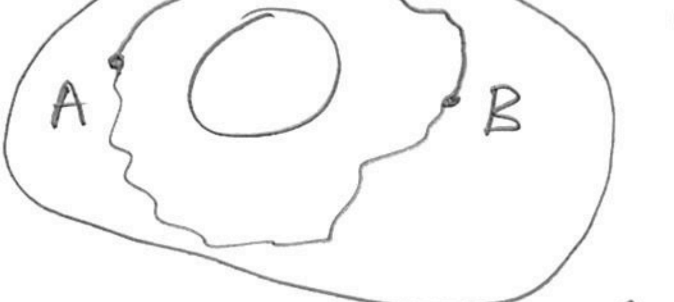
In topology, a topological space is called simply connected ~~it it is~~ every path between two points can be continously transformed into any other path, while preserving the two end points

מרחב קשיר

(plane with a hole in it)

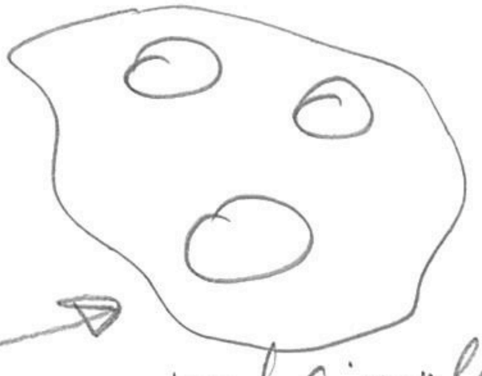
מרחב פשוט קשיר





a simply connected domain

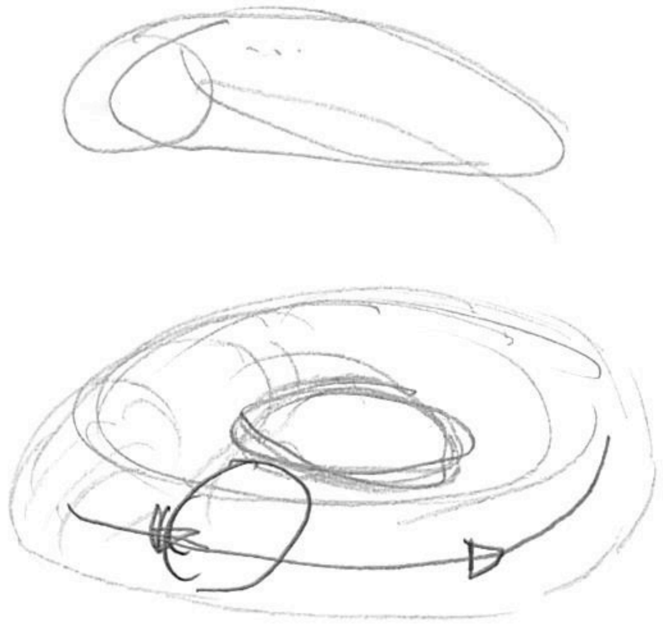
not simply connected domain



a sphere (simply connected)

מרחב קשיר

not simply connected



Torus

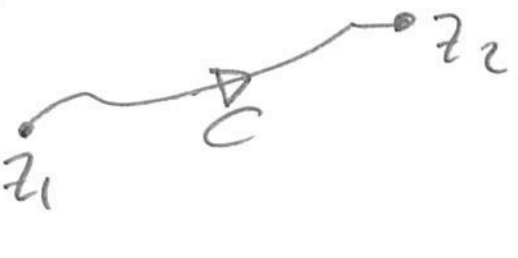
another example of non simply connected space

Let D be a simply connected open set. Let f be an analytic function in D (differentiable, singlevalued).

Then the integral

$$\int_C dz\, f(z)$$

does not depend on C, as long as its end points are fixed.



In particular

$$\oint_C f(z)\,dz = 0$$

C

if the curve C is closed.

<u>Proof</u>

$$\int_C dz\, f(z) = \int_C [u\,dx - v\,dy] + i\int_C [u\,dy + v\,dx]$$

Consider the real part

$$\int_C u\,dx - v\,dy$$ for it to

be independent on the trajectory

$$\underbrace{u\,dx - v\,dy}_{\text{must be a differential}} = dw$$

$$dw(x,y) = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy$$

$$u = \frac{\partial w}{\partial x} \qquad v = -\frac{\partial w}{\partial y}$$

$$\cancel{\frac{\partial u}{\partial x} = \frac{\partial^2 w}{\partial x \partial y} = -\frac{\partial v}{\partial y}}$$

$$\cancel{\frac{\partial u}{\partial y} = \partial^2}$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{\partial^2 w}{\partial x \partial y} = -\frac{\partial v}{\partial x}}$$

Similarly

$$u\,dy + v\,dx = d\tilde{w}$$

$$d\tilde{w} = \frac{\partial \bar{w}}{\partial x}\,dx + \frac{d\bar{w}}{\partial y}\,dy$$

$$\frac{\partial \bar{w}}{\partial x} = v \qquad \frac{\partial \bar{w}}{\partial y} = u$$

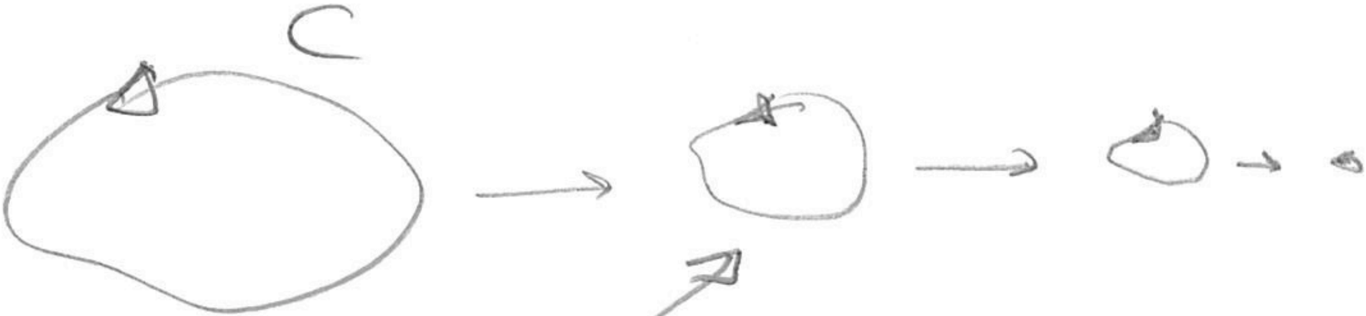
$$\frac{\partial \bar{w}}{\partial x \partial y} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} \end{aligned}}$$

but these are Caushy-Riemann cond. for f to be analytic in D

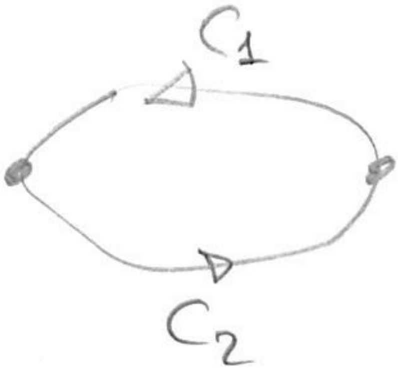
Therefore we proofed the theorem ■

The theorem for the closed curve follows trivialy.



Proof by picture

More formal



$$\oint_C dz f(z) = \int_{C_1} dz f(z) + \int_{C_2} dz f(z) =$$

$$= \int_{C_1} dz f(z) - \int_{C_2^-} dz f(z)$$

$$\int_{C_2^-} dz f(z) = \int_{C_1} dz f(z)$$

because C_2^- has the same endpoints and is oriented in the same way as C_1

Alteranive proof

$$dz = dx + i\,dy$$
$$f = u + iv$$
$$df = du + i\,dv$$

$$\oint_C dz\, f(z) = \oint (dx + i\,dy)\cdot(u + iv) =$$
$$= \oint [dx\, u - v\, dy] + i[dy\, u + v\, dx]$$

Green's theorem

$$\oint_C L\,dx + M\,dy = \int_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$

$$\|$$

$$\oint_{\partial D} \omega = \int_D d\omega$$

$$\omega = L\,dx + M\,dy$$
$$d\omega = \frac{\partial L}{\partial x}\cancel{dx\wedge dx} + \frac{\partial L}{\partial y}dy\wedge dx + \frac{\partial M}{\partial x}dx\wedge dy + \frac{\partial M}{\partial y}\cancel{dy\wedge dy} =$$
$$= \frac{\partial L}{\partial y}dy\wedge dx + \frac{\partial M}{\partial x}dx\wedge dy$$
$$= \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\wedge dy$$

$$\int dz\, f(z) =$$

$$= \oint_C [u\,dx - v\,dy] + i\,[v\,dx + u\,dy] =$$

$$= \int_D dx \wedge dy \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) =$$

$$= + \int dx \wedge dy \left\{ -\underbrace{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}_{= 0} + i\underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{= 0} \right\} =$$

$$= 0$$

Now consider a ~~not~~ a simply connected domain.

$f(z) = 1/z$ this function is singular at $z=0$

Exclude the point $z=0$ from the domain. The domain with a point excluded is no longer a simply connected

Compute the integral on the unit circle

$$\oint \frac{dz}{z} =$$

$z = Re^{i\varphi}$

$R=1$

$dz = iRe^{i\varphi}d\varphi$

$$= \int_0^{2\pi} d\varphi \, \frac{i\cancel{R}\cancel{e^{i\varphi}}}{\cancel{R}\cancel{e^{i\varphi}}} =$$

$$= i\int_0^{2\pi} d\varphi = 2\pi i$$

Note that R drops out of calculation, so the integral on any circle $= 2\pi i$.

Moreover, the integral on any contour that encompases the origin once is $2\pi i$.

Note that the integral of 1 along the unit circle

$$\oint_{|z|=1} dz = \int_0^{2\pi} iRe^{i\varphi} d\varphi = iR\int_0^{2\pi} e^{i\varphi} d\varphi =$$

$$= iR \left.\frac{e^{i\varphi}}{i}\right|_0^{2\pi} = R\left(e^{i2\pi} - e^{i0}\right) = 0$$

Indeed, for the function $f(z) = 1$ the point of origin is regular and we could shrink the contur of integration.

Less trivial example:

$$f(z) = \frac{1}{z^2}$$

$$\oint_C dz\, f(z) = \oint_C \frac{dz}{z^2} = \int_0^{2\pi} d\varphi \frac{iRe^{i\varphi}}{R^2 e^{2i\varphi}} =$$

$$= \frac{i}{R}\int_0^{2\pi} d\varphi\, e^{-i\varphi} = \frac{i}{R}\left(\frac{1}{-i}\right) \left. e^{-i\varphi}\right|_0^{2\pi} =$$

$$= -\frac{1}{R}\left(e^{-2\pi i} - e^0\right) = 0$$

Note tha $1/z^2$ is singular at $z=0$

Cauchy's integral formula

Let D be an open subset of the complex plane $\mathbb{C}$, let C be an oriented ~~circle~~ boundary (boundary of the domain)

f analytic in D. For any point $\omega \in D$

$$f(\omega) = \frac{1}{2\pi i} \oint \frac{f(z)\,dz}{z-\omega}$$

Simple example $\quad \omega = 0$
$\qquad\qquad\qquad\quad f(z) = 1$

$$1 = \frac{1}{2\pi i} \oint \frac{dz}{z}$$ as we have seen alredy