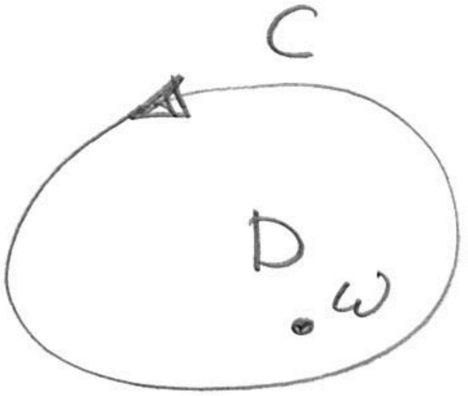
Cauchy's integral formula

Lecture 4

Suppose D is a simple connected domain, f analytic with D and C counterclockwise boundary of D



ω is a point on the complex plane in D

$$f(\omega) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-\omega}\, dz$$

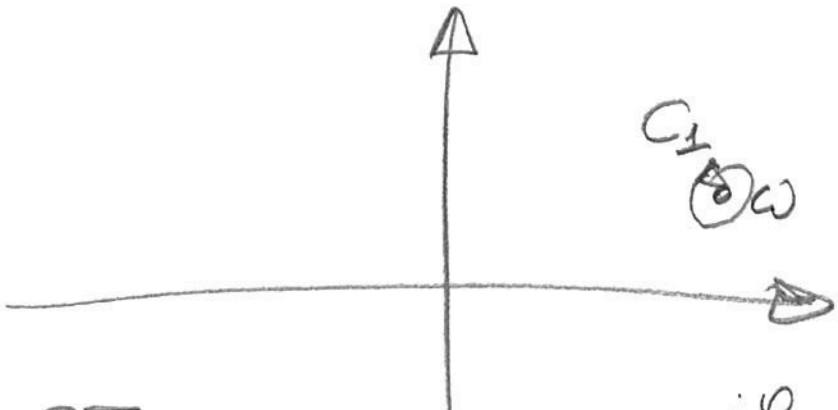
"Proof" We know that if function is analytic the integral $\oint_C dz\, g(z)$ does not depend on C

$$g(z) \equiv \frac{f(z)}{z-\omega}$$

Let us shrink C to C_1 a small circle around ω

$$z = \omega + \varepsilon e^{i\varphi}$$

$$\varepsilon \to 0 \qquad 0 < \varphi < 2\pi$$



$$f(\omega) = \frac{\lim_{\varepsilon \to 0}}{2\pi i} \oint_{C_1}^{2\pi} \frac{f(\omega + \varepsilon e^{i\varphi}) i \varepsilon e^{i\varphi} \, d\varphi}{\varepsilon e^{i\varphi} + \cancel{\omega} - \cancel{\omega}} =$$

$$= \frac{1}{2\pi i} f(\omega) \int_0^{2\pi} i \, d\varphi = f(\omega)$$

In other words

$$f(\omega) = \frac{1}{2\pi i} f(\omega) \oint_{C_1} \frac{dz}{z} = f(\omega)$$

$\boxed{E}$

Let us differentiate

$$\frac{d}{d\omega} f(\omega) = f'(\omega) = \frac{1}{2\pi i} \oint_C \left(\frac{d}{d\omega} \frac{f(z)}{z-\omega} \right) dz$$

$$= \frac{1}{2\pi i} \oint_C f(z) dz \left(\frac{d}{d\omega} \frac{1}{z-\omega} \right) =$$

$$= \frac{1}{2\pi i} \oint \frac{f(z) dz}{(z-\omega)^2}$$

$$f'(\omega) = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z-\omega)^2}$$

By differentiating n times

$$f^{(n)}(\omega) = \frac{n!}{2\pi i} \oint_C \frac{f(z) dz}{(z-\omega)^{n+1}}$$

Taylor series for analytic function f

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(\omega)}{n!}(z-\omega)^n$$

Examples:

$$e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots$$

$$\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}+\dots$$

$$\operatorname{sh} z = z+\frac{z^3}{3}+\frac{z^5}{5!}+\dots$$

converges for all z

for the branch $\ln(1)=0$

$$\ln(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}+\dots$$

$$(1+z)^\alpha = 1+\alpha z+\frac{\alpha(\alpha-1)}{2!}z^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3+\dots$$

converges for $|z|<1$

for the branch $(1)^\alpha = 1$

Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-\omega)^n$$

this is a generalization of the Taylor series.

Examples of Laurent series:

- $f(z) = \dfrac{1}{z-\omega}$

  $a_{-1} = 1$ all other $a_n = 0$

  $f(z)$ has a simple pole at w

- $f(z) = \dfrac{\sin z}{z^2} =$

$$= \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right) =$$

$$= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} + \dots$$

$a_{-1} = 1$ $\quad a_0 = 1$ $\quad a_1 = -\frac{1}{3!}$ $\quad a_2 = 0$ $\quad a_3 = \frac{1}{5!}$ ...

Let us apply Cauchy' integral formula for analytic function

$$f(z) = f(\omega) + f'(\omega)(z-\omega) + \frac{1}{2} f''(\omega)(z-\omega)^2$$

$$f(\omega) = \frac{1}{2\pi i} \oint_C \frac{dz}{z-\omega} \left\{ f(\omega) + f'(\omega)(z-\omega) + \frac{1}{2} f''(\omega)(z-\omega)^2 + \dots \right\}$$

$$= \frac{f(\omega)}{2\pi i} \oint \frac{dz}{z-\omega} +$$

$$+ \frac{f'(\omega)}{2\pi i} \oint \frac{dz}{z-\omega}(z-\omega) + \frac{f''(\omega)}{2\pi i \cdot 2} \oint \frac{dz\,(z-\omega)^2}{z-\omega}$$

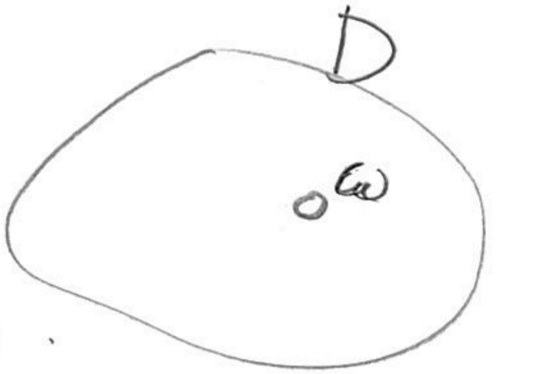
$$+ \dots = \overbrace{\frac{f(\omega)}{2\pi i} \oint \frac{dz}{z-\omega}}^{1} + \frac{f'(\omega)}{2\pi i} \underbrace{\oint dz}_{0}$$

$$+ \frac{f''(\omega)}{4\pi i} \underbrace{\oint dz\,(z-\omega)}_{0} + \dots =$$

$$= f(\omega)$$

As expeded, analytic function $f(z)$ can be expanded into Taylor series and satisfy Cauchy's integral formulae.

Let us now take a function that has Laurent expansion, rather than Taylor expansion near point ω.
It means that the function is analytic in the domain D with point ω excluded from it (domain is not simple connected).



In this case Cauchy's integral formulae does not apply.
Nevertheless, let us take and apply the transformation that appears in that formula for the function f and compute

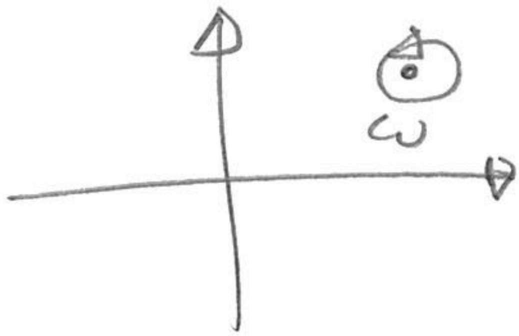
$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\omega}\,dz = \frac{1}{2\pi i}\oint \frac{dz}{z-\omega} \cdot$$

$$\cdot \sum_{n=-\infty}^{\infty} a_n (z-\omega)^n$$

$$= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \oint_C dz \, \frac{a_n}{z-\omega} (z-\omega)^n =$$

$$= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} a_n \oint_C dz \, (z-\omega)^{n-1}$$

We have encountered such integrals before.



Parametrise

$$z = \omega + \varepsilon e^{i\varphi}$$

$$dz = \varepsilon i e^{i\varphi} d\varphi$$

$$\oint_C dz \, (z-\omega)^{n-1} = \int_0^{2\pi} d\varphi \left(\varepsilon i e^{i\varphi}\right) \left(\varepsilon e^{i\varphi}\right)^{n-1} =$$

$$= \int_0^{2\pi} d\varphi \; \varepsilon \, \varepsilon^{n-1} \, i \, e^{i\varphi(\cancel{1}+n-\cancel{1})} =$$

$$= \varepsilon^n i \int_0^{2\pi} d\varphi \; e^{i\varphi n} = i 2\pi \, \delta_{n,0} .$$

We see that

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\omega}\,dz = \frac{1}{2\pi i}\,a_0\cdot 2\pi i =$$

$$= a_0 \neq f(\omega)$$

Let us now repeat this excercise and compute

$$\frac{1}{2\pi i}\oint_C dz\, f(z) =$$

$$= \frac{1}{2\pi i}\oint_C dz \sum_{n=-\infty}^{\infty} a_n (z-\omega)^n =$$

$$= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} a_n \oint_C dz\,(z-\omega)^n$$

$$\oint_C dz\,(z-\omega)^n = \int_0^{2\pi} d\varphi\,(\varepsilon i e^{i\varphi})(\varepsilon e^{i\varphi})^n =$$

$$= \varepsilon^{1+n}\, i \int_0^{\pi} d\varphi\, e^{i\varphi(n+1)} = 2\pi i\,\delta_{n,-1}$$

Therefore

$$\frac{1}{2\pi i}\oint_C dz\, f(z) = \frac{2\pi i}{2\pi i}\sum_{n=-\infty}^{\infty} a_n\, \delta_{n,-1} =$$

$$= \frac{2\pi i}{2\pi i}\, a_{-1} = a_{-1}$$

residue of the function $f(z)$ at ω

The residue theorem

Let f be analitic within domain D, with exception of a finite number of points at which $f(z)$ has poles

$$\oint_C dz\, f(z) = \sum_i \oint_{C_i} f(z)\, dz =$$

$$= \sum_{\text{poles}} 2\pi i \left(\text{residue at the pole } i\right)$$