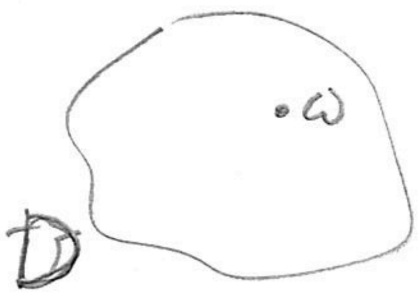
Points of singularities , Lectures

Isolated singularity

D is an open subset of the complex plane.



f is a complex differentiable function for all z in $D\setminus\{\omega\}$ [exluding ω]

1) If there is a finite limit $\lim\limits_{z\to\omega} f(z)$ then ω is a removable singularity.

   Example $\quad f(z) = \dfrac{\sin z}{z}$

$$\lim_{z\to 0} f(z) = 1$$

2) The point ω is a ~~point~~ pole if there is an infinite limit $\lim\limits_{z\to\infty} f(z) \to \infty$

   Examples $\quad f(z) = \dfrac{z}{(z-1)^2}$, $f(z) = \dfrac{1}{z^3}$, ...

3.) The point ω is essential singularity if there is no limit $\lim_{z \to \omega} f(z)$.

Example:

$$f(z) = e^{1/z}$$

Noninsolated singularities

To be discussed later

Laurent series for different types of isolated singularities

1) Removable singularity

$$\frac{\sin z}{z} \simeq 1 - \frac{z^2}{6} + \frac{z^4}{120} + \dots$$

In other words this function is Taylor expandable and in Laurent series

$$f(z) = \underbrace{\dots + \frac{c_{-n}}{(z-\omega)^n} + \dots \frac{c_{-1}}{z-\omega}}_{\overset{\shortparallel}{0}} + c_0 + c_1(z-\omega) + \dots$$

all term with negative power of $(z-\omega)$ are zero

2) Non-essential singularities (Poles)

$$f(z) = \frac{z}{(z-1)^2} = \frac{z-1+1}{(z-1)^2} = \frac{1}{(z-1)^2} + \frac{1}{z-1}$$

conclusion: there is a finite number of terms with negative powers of $(z-\omega)$

In general, for the pole of degree n

$\lim\limits_{z\to\omega} (z-\omega)^n f(z) = c_n$ exist and

$$f(z) = \underbrace{\frac{c_{-n}}{(z-\omega)^n} + \dots \frac{c_{-1}}{z-\omega}}_{\text{a finite number of terms}} + \underbrace{c_0 + c_1(z-\omega) + \dots}_{\sum_{n=0}^{\infty} c_n (z-\omega)^n}$$

may be a finite or an infinite number

3) Essential singularities

Consider an example

$$f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \dots \frac{1}{n!\,z^n} + \dots$$

$$= 1 + \sum_{n=-1}^{-\infty} z^{-n} \frac{1}{n!}$$

There are infinitelly many terms with negative powers of $(z-\omega)$.

In the vicinity of an essential singularity function takes all possible values and one may choose a set of points z_k, such that $z_k \to \omega$ for $k \to \infty$ that $f(z_k) \to A$ [A is any number in the complex plane, including ∞].

Example

$$f(z) = e^{1/z} = \sum_{n=0}^{-\infty} \frac{z^{-n}}{n!}$$

choose $A = \infty$

$$z_k = \frac{1}{k}$$

$$f(z_k) = e^{k} \xrightarrow[k \to \infty]{} \infty$$

choose $A = 0$

$$z_k = -\frac{1}{k}$$

$$f(z_k) = e^{-k} \xrightarrow[k \to \infty]{} 0$$

choose $A \neq 0, \neq \infty \to z_k = \dfrac{1}{2\pi k i + \ln A}$

any branch of ln

$$f(z_k) = e^{2\pi i k + \ln A} = A$$

Let us discuss the connection between singularity at point ω and Cauchy theorem.

C

consider an integral along C, that encompasses ω and function f, such that ω is a singular point. Assume that this is the only singular point inside.

the only singular point inside.

If ω is a removable singularity we shrink C into an infinitesimal loop

$$\oint_C dz\, f(z) = \oint_C dz\, f(\omega) = f(\omega) \oint_C dz = 0$$

therefore $\text{res}\, f(\omega) = 0$ and removable singularities do not affect the value of an integral.

Next we discuss the res of the function that has a pole of degree n

$$f(z) = \frac{c_{-n}}{(z-\omega)^n} + \dots \frac{c_{-1}}{z-\omega} + c_0 + c_1(z-\omega) + \dots$$

How do we find c_{-1}?

Multiply $f(z)$ by $(z-\omega)^n$

$$f(z)(z-\omega)^n = c_{-n} + (z-\omega)\,c_{-n+1} + (z-\omega)^2 c_{-n+2} + \dots + c_{-2}(z-\omega)^{n-2} + c_{-1}(z-\omega)^{n-1} + c_0 (z-\omega)^n + \dots$$

Differentiate with respect to z $(n-1)$ times

Terms with $n<-1$ are polinomials with degree $< n-2 \to$ drop after $n-1$ derivatives

$$\frac{d^{n-1} f(z)}{dz^{n-1}} = (n-1)!\,c_{-1} + c_0 n!\,(z-\omega) + \dots$$

$$\left.\frac{d^{n-1} f(z)}{dz^{n-1}}\right|_{z=\omega} = (n-1)!\,c_{-1} \to$$

$$c_{-1} = \frac{1}{(n-1)!} \left.\frac{d^{n-1} f(z)}{dz^{n-1}}\right|_{z=\omega}$$

For the simple pole

$$\operatorname{res} f(\omega) = \left.(z-\omega) f(z)\right|_{z=\omega} \equiv \lim_{z\to\omega}\{(z-\omega) f(z)\}$$

limit $z \to \omega$

In particular if

$$f(z) = \frac{\varphi(z)}{g(z)}$$, where $\varphi(z), g(z)$ are are analytic function in D

$$g(\omega) = 0 \, , \quad \varphi(\omega) \neq 0$$

$$\operatorname{res} f(z) = (z-\omega) f(\omega) \Big|_{z=\omega} =$$

$$= \lim_{z\to\omega} \frac{(z-\omega)\varphi(z)}{g(z)} = \lim_{z\to\infty} \frac{\cancel{(z-\omega)}\varphi(\omega)}{\cancel{(z-\omega)}g'(\omega)} = \frac{\varphi(\omega)}{g'(\omega)}$$

expand $\varphi(z) = \varphi(\omega)$

$$g(z) = g(\omega) + (z-\omega)g'(\omega) = (z-\omega)g'(\omega)$$

Example

$\omega = 0$

$$f(z) = \frac{\cos z - 2}{\sin z}$$

$$\operatorname{res} f(0) = \frac{\cos 0 - 2}{(\sin z)'\big|_{z=0}} = \frac{-1}{\cos 0} = -1$$

Example

$$f(z) = \mathrm{ctg}\, z = \frac{\cos z}{\sin z}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = 0$$

$$e^{iz} = e^{-iz} \rightarrow e^{2iz} = 1$$

$$2iz = 2\pi i k$$

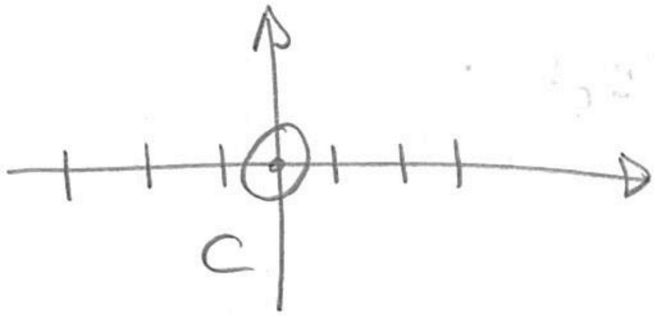
$$z = \pi k$$

$$\mathrm{res}\{f(z_k)\} = \frac{\cos z_k}{(\sin z)'_{z_k}} = \frac{\cos z_k}{\cos z_k} = 1$$

Therefore

$$\oint_C \frac{\cos z}{\sin z}\, dz = 2\pi i$$

$|z| < \frac{\pi}{2}$



Example

$$f(z) = \mathrm{cth}\, z = \frac{\cosh z}{\sinh z}$$

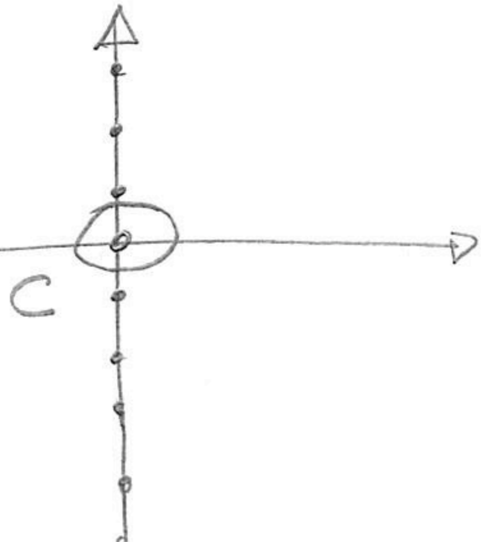
$f(z_k) = 0 \qquad \sinh z = \dfrac{e^{z} - e^{-z}}{2} = 0$

$e^{z} = e^{-z} \quad \rightarrow e^{2z} = 1$

$2z = 2\pi i k$

$z = \pi i k$

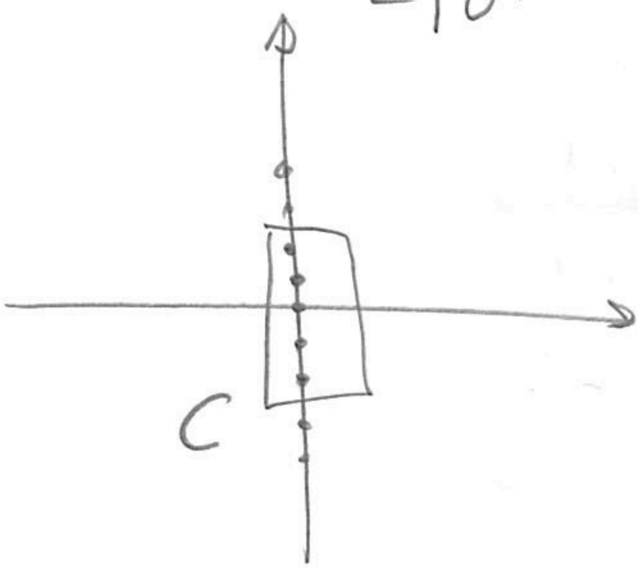
$\mathrm{res}\, f(z) = \dfrac{\cosh z_k}{\cosh z_k} = 1$



Integral over C $\qquad |z| < \pi$

$$\oint_C dz\, \mathrm{ch} z = 2\pi i$$

Note that if C is bigger



$$\oint_C dz \coth z = 2\pi i \times \left[\text{number of } \cancel{\text{point}} \text{ poles inside } C\right]$$

If we integrate function $f(z)$ that is analytic inside C, and does not vanish at $z_k = \pi i k$

$$\boxed{\oint_C dz \coth z \, f(z) = 2\pi i \sum_{k \text{ inside } C} f(z_k)}$$

Consider an analytic function $f(z)$ in D.

and $\frac{d}{dz} \ln f(z) = \frac{f'(z)}{f(z)}$

Assume that $f(a) = 0$ and it is a zero of the order 1, i.e

$f(z) \cong c_1 (z-a) + c_2 (z-a)^2 + \dots$

The integral around a

$$\oint_C dz \frac{d \ln f(z)}{dz} = \oint_C dz \frac{\cancel{c_1}}{\cancel{c_1}(z-a)} = 2\pi i$$

Therefore if $f(z)$ has a zero of the order

$1 \rightarrow \frac{d}{dz} \ln f(z) \simeq \frac{1}{z-a} \rightarrow \text{Res} = 1$

$\left[\text{res}\left\{ \frac{d \ln f}{dz} (a) \right\} = 1 \right.$

Now generalize

zero of the order n

$f(z) \cong c_n (z-a)^n + c_{n+1} (z-a)^{n+1} + \dots$

$$\frac{d \ln f}{dz} \cong \frac{n\, c_n (z-a)^{n-1}}{c_n (z-a)^n} = \frac{n}{z-a} \rightarrow \text{Res} = n$$

Next we consider the function f with poles.

$$f(z) = \frac{C_{-1}}{z-\omega}$$

$$\frac{d}{dz} \ln f = \frac{f'}{f} = -\frac{C_{-1}}{(z-\omega)^2} \frac{z-\omega}{C_{-1}} = -\frac{1}{z-\omega}$$

$$\mathrm{Res}\left\{\ln f\right\} = -1$$

$$f(z) = \frac{C_{-n}}{(z-\omega)^n}$$

$$\frac{d}{dz} \ln f = \frac{f'}{f} = \frac{-n\, C_{-n}}{(z-\omega)^{n+1}} \frac{(z-\omega)^n}{C_{-n}} = \frac{-n}{z-\omega}$$

$$\mathrm{Res} \ln f = -n$$

Therefore pole of the order n in function f $\longrightarrow$ pole of the order 1 of function $\frac{d\ln f}{dz}$ with $\mathrm{Res}\left\{\frac{d\ln f}{dz}\right\} = -n$

Zero of the order n in f $\rightarrow$ pole of the order 1 with $\mathrm{Re}\left\{\frac{d\ln f}{dz}\right\} = n$

Theorem

Let $f(z)$ be analytic function in D, except a finite number of poles at points

b_1 of the order p_1

b_2 — " — p_2

...

b_m — " — p_m

and it has zeroes at

a_1 of the order n_1

a_2 — " — n_2

...

a_ℓ — " — n_ℓ

zeroes and poles are inside D [not at the boundary].

From derivation above

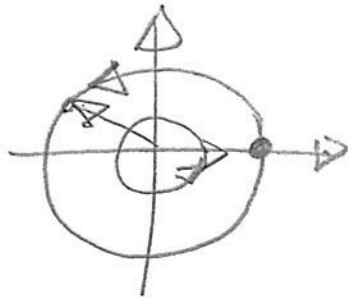
$$\frac{1}{2\pi i}\oint_C dz\,\frac{f'(z)}{f(z)} = (n_1+n_2+\dots n_\ell)-(p_1+p_2+p_m) \equiv N-P$$

$$\frac{1}{2\pi i}\oint_C d\ln f(z) = \frac{1}{2\pi i}\oint d\left(\ln|f(z)| + i\,\mathrm{Arg} f(z)\right) =$$

$$= \frac{1}{2\pi}\oint_C d\,\mathrm{Arg} f(z) = \underbrace{\frac{1}{2\pi}\Delta \mathrm{Arg} f(z)}_{\text{number of times vector } \begin{pmatrix}\mathrm{Re} f(z)\\ \mathrm{Im} f(z)\end{pmatrix} \text{ turns around origin.}}$$

Example

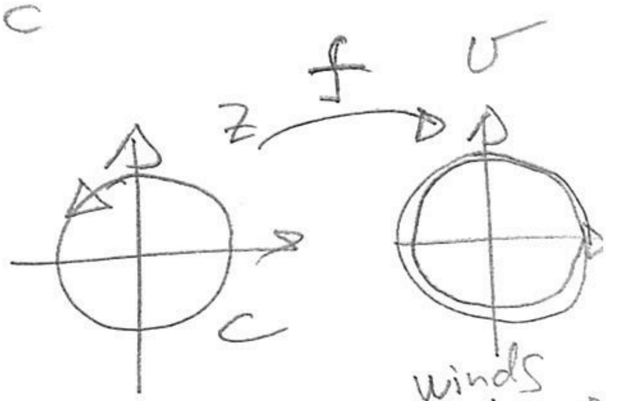
$$f(z) = z$$



$$\frac{d\ln f}{dz} = \frac{f'}{f} = \frac{1}{z}$$

$$\frac{1}{2\pi i}\oint_C dz\, d\ln f = \frac{1}{2\pi i}\oint_C \frac{dz}{z} = \underline{1}$$

Example



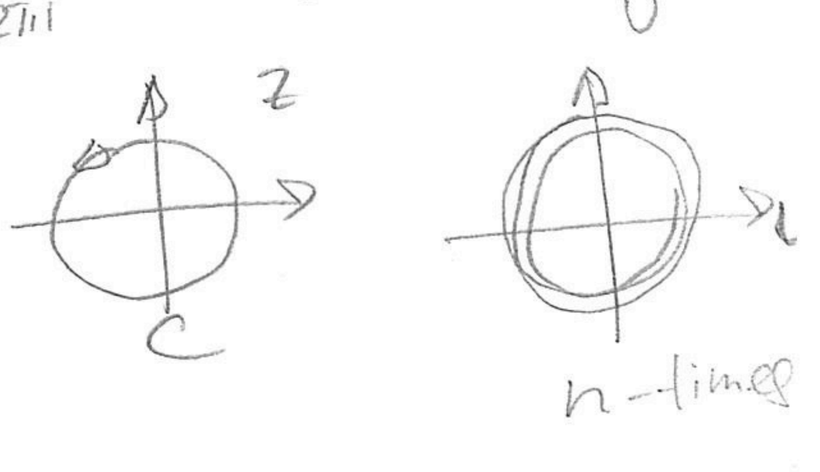
$$f(z) = z^2$$

$$\frac{d\ln f}{dz} = \frac{2}{z}$$

$$\frac{1}{2\pi i}\oint_C dz\, \frac{f'}{f} = \frac{1}{2\pi i}\oint dz\,\frac{2}{z} = 2$$

Example

$$f(z) = z^n$$



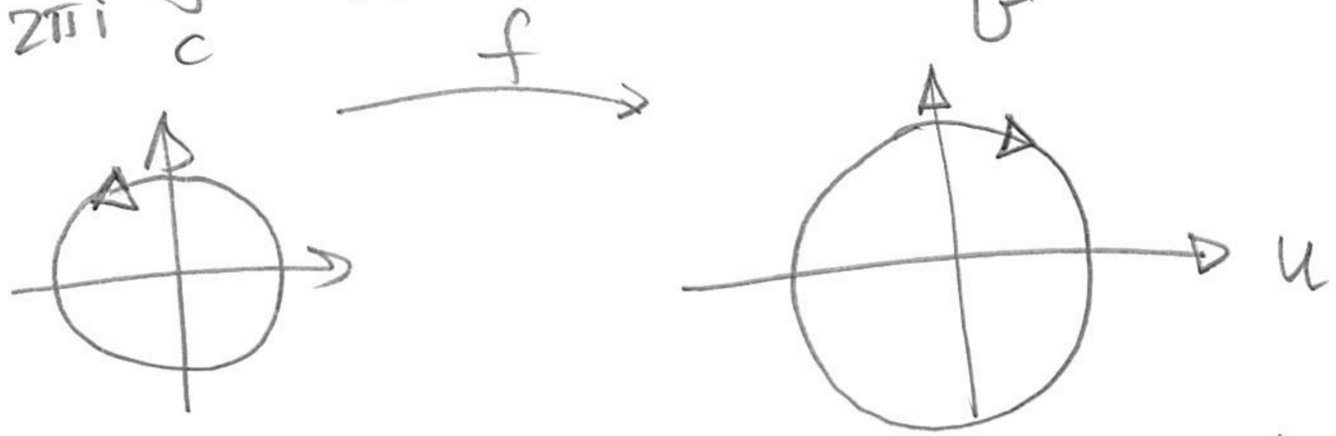
$$\frac{1}{2\pi i}\oint dz\, \frac{f'}{f} = n$$

Example

$$f(z) = 1/z$$

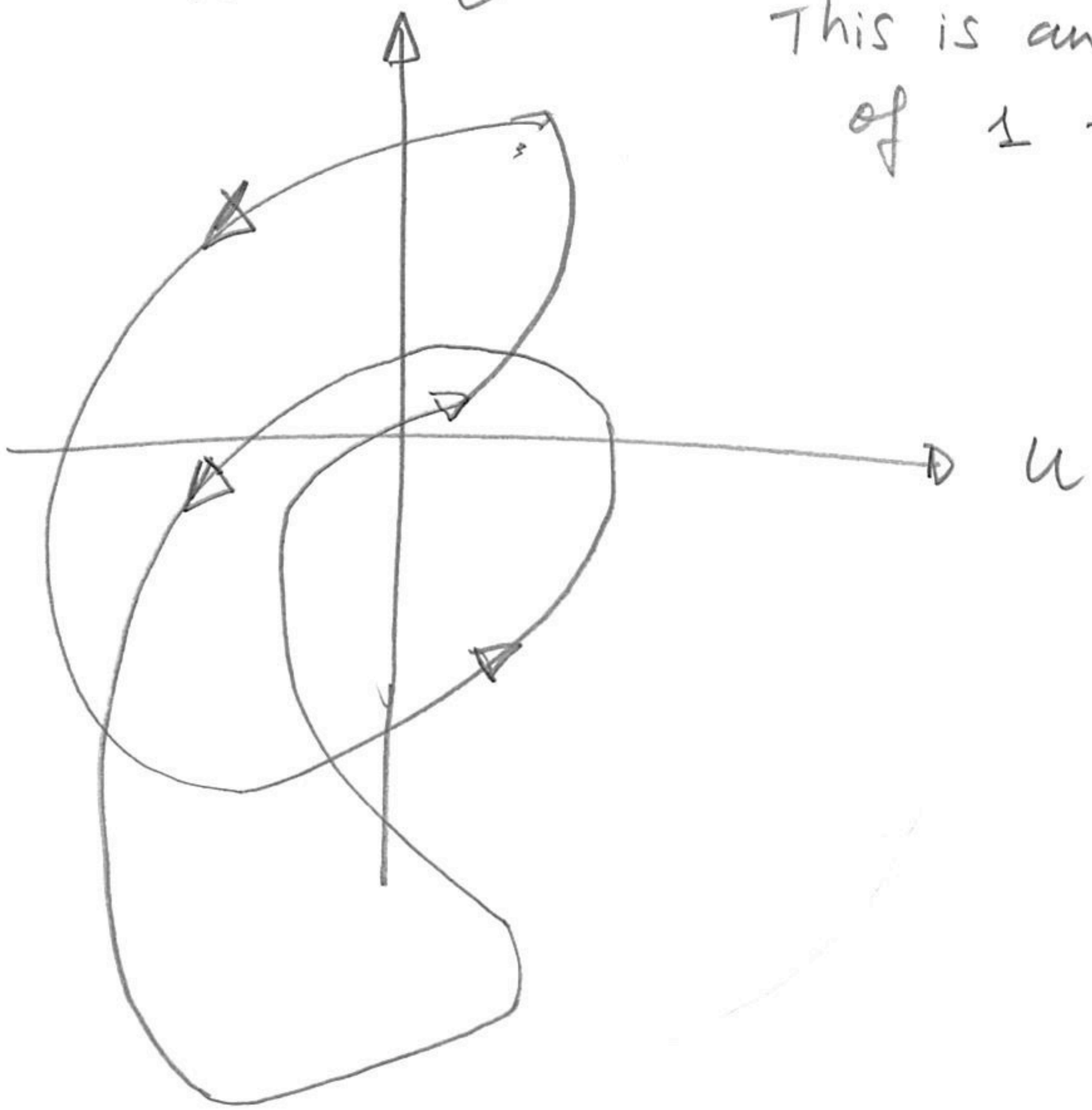
$$\frac{d}{dz} \ln f = -\frac{1}{z}$$

$$\frac{1}{2\pi i} \oint_C dz \frac{d}{dz} \ln f = -1$$

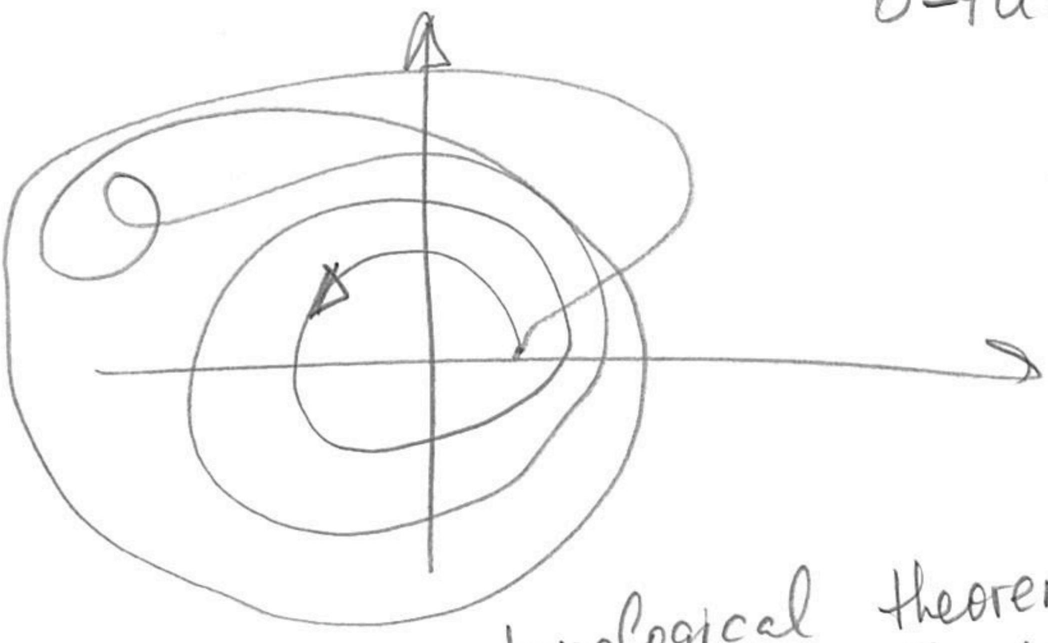


turns in opposite (clockwise direction)

This is an example of 1 turn



0-turn



This is a topological theorem that relates zeroes and poles of the function $f(z)$ the topological index [number of turns]