Rouche's theorem

Let $f(z)$ and $g(z)$ be analytic within a simple closed contour C on the complex plane.
Suppose $|f(z)| > |g(z)|$ everywhere on C.
Then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.

The proof

Number of poles $= 0$ [f and g are analytic and have no poles]

$$\text{Number of zeros}_{f+g} - \underbrace{\text{Number of Poles}_{f+g}}_{0} =$$

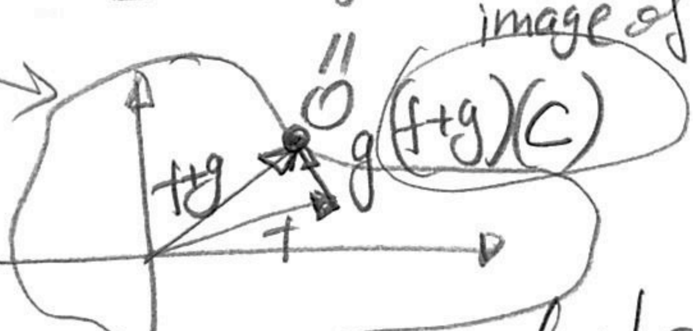
$$= \frac{1}{2\pi} \Delta_C \arg\left(f(z)+g(z)\right) =$$

$$= \frac{1}{2\pi i} \Delta_C \ln\left(f(z)+g(z)\right)$$

$$\text{Number of zeros}_{f+g} = \frac{1}{2\pi i} \Delta_C \ln\left(f\left(1+\frac{g}{f}\right)\right) =$$

$$= \frac{1}{2\pi i}\Delta_C \ln f + \frac{1}{2\pi i}\Delta_C \underbrace{\ln\left(1+\frac{g}{f}\right)}_{0} =$$

dog on the leash and the owner and the pole



$$= \frac{1}{2\pi i}\Delta_C \ln f \qquad \blacksquare$$

The fundamental theorem of algebra immediately follows from this.

The polynomial of n'th degree

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$$

has exactly n zeroes, i.e.

$P(z_k) = 0$ hold for n points including the multiplicity of the point.

To proove this denote $f(z) = a_n z^n$

$$g(z) = a_{n-1} z^{n-1} + \dots a_0$$

and choose a circle of a radius $R \gg 1$ such that $|f| > |g|$. Note that neither f nor g have poles inside the circle.

Therefore $P(z)$ has exactly the same ammount of zeroes as $a_n z^n$, that is n. $\blacksquare$

Example. Let us use Rouche's theorem to determine the number of zeroes inside the circle of a radius R on the complex plane.

$$P(z) = z^4 + z^2 + 1$$

$$|z^4| = |z^2+1| = \sqrt{(z^2+1)\overline{(z^2+1)}} =$$

$$= \sqrt{|z|^4 + z^2 + \bar{z}^2 + 1}$$

$$z = Re^{i\varphi}$$

$$R^4 = \sqrt{R^4 + 1 + 2R^2\cos 2\varphi}$$

$$R^8 \overset{!}{>} R^4 + 1 + 2R^2 \cos 2\varphi \quad \text{for all } \varphi$$

$$R^8 > R^4 + 2R^2 + 1$$

$$R^8 > (R^2+1)^2$$

$$R^4 > R^2 + 1$$

$$R^4 - R^2 - 1 > 0$$

$$R_{1,2} = \frac{1}{2}\left[1 \pm \sqrt{1+4}\right] = \frac{1}{2}\left[1 \pm \sqrt{5}\right]$$

$$R = \frac{1+\sqrt{5}}{2}$$

Conclusion → 4 roots must lie within the circle $\frac{1+\sqrt{5}}{2} \simeq 1.6$

Check explicitly.

Denote $\zeta = z^2$

$$P(\zeta) = \zeta^2 + \zeta + 1 = 0$$

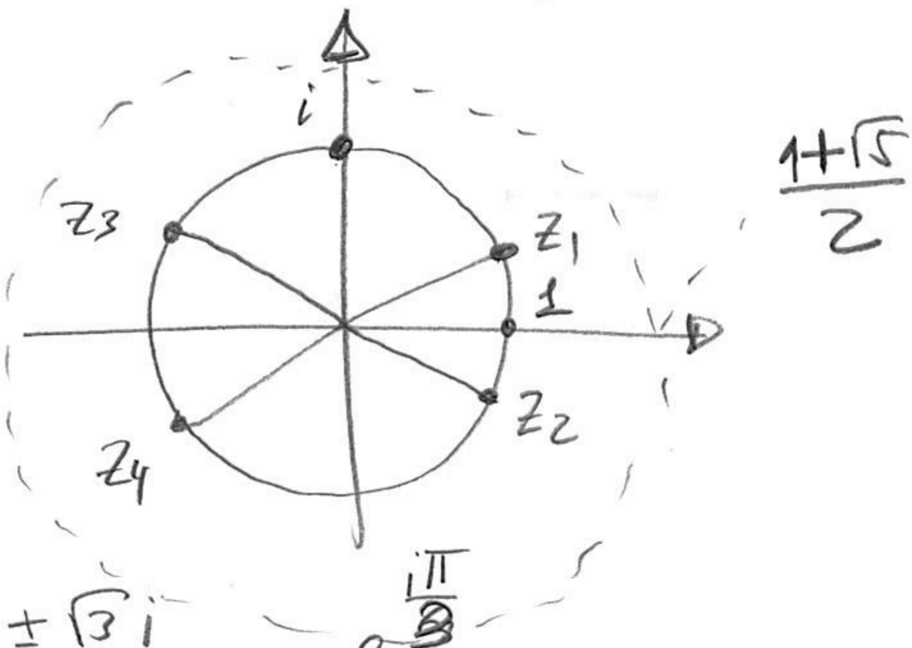
$$\zeta_{1,2} = \frac{1}{2}\left[-1 \pm \sqrt{1-4}\right] =$$

$$= \frac{1}{2}\left[-1 \pm \sqrt{-3}\right] = \frac{-1 \pm \sqrt{3}\,i}{2}$$

$$z = \pm\sqrt{\frac{-1 \pm \sqrt{3}\,i}{2}}$$

$$|z| = \left(\frac{(-1+\sqrt{3}\,i)(-1-\sqrt{3}\,i)}{4}\right)^{1/4} = \left(\frac{1+\sqrt{3}\,i-\sqrt{3}\,i+3}{4}\right)^{1/4}$$

$$= 1$$



$$\frac{-1\pm\sqrt{3}\,i}{2} = e^{\frac{i\pi}{3}}$$

$$\zeta = e^{\pm i\pi/3} \qquad z = \pm e^{\pm\frac{i\pi}{6}}$$

Result agree with Rouche's theorem that all zeroes lie on the unit circle, that is inside the circle of the radius R that we have $R = \frac{1+\sqrt{5}}{2}$. Note that R found from Rouche's theorem is an upper bound.

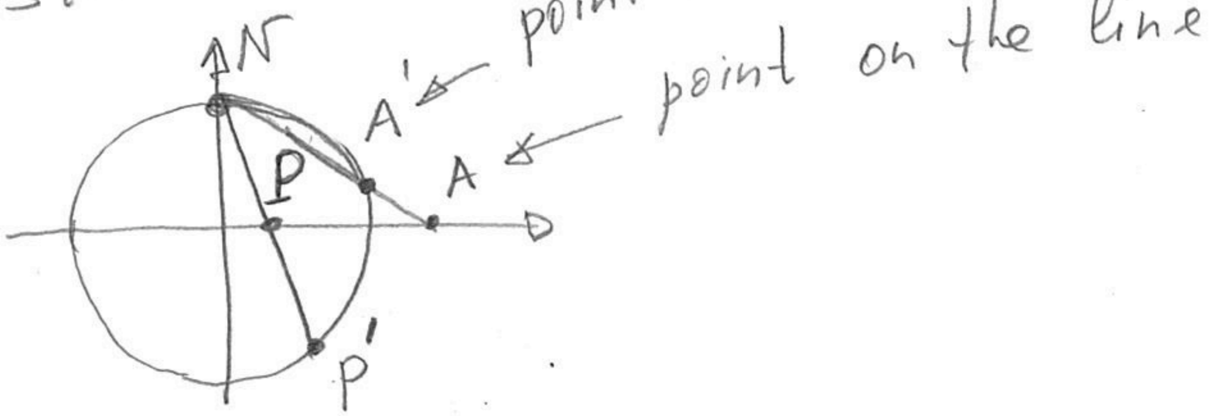
Riemann sphere

The idea is to augment the complex plane by adding an additional point to the complex plane, that is called infinity. This construction $\mathbb{C} \cup \infty$ is called an extended complex plane, or Riemann surface.

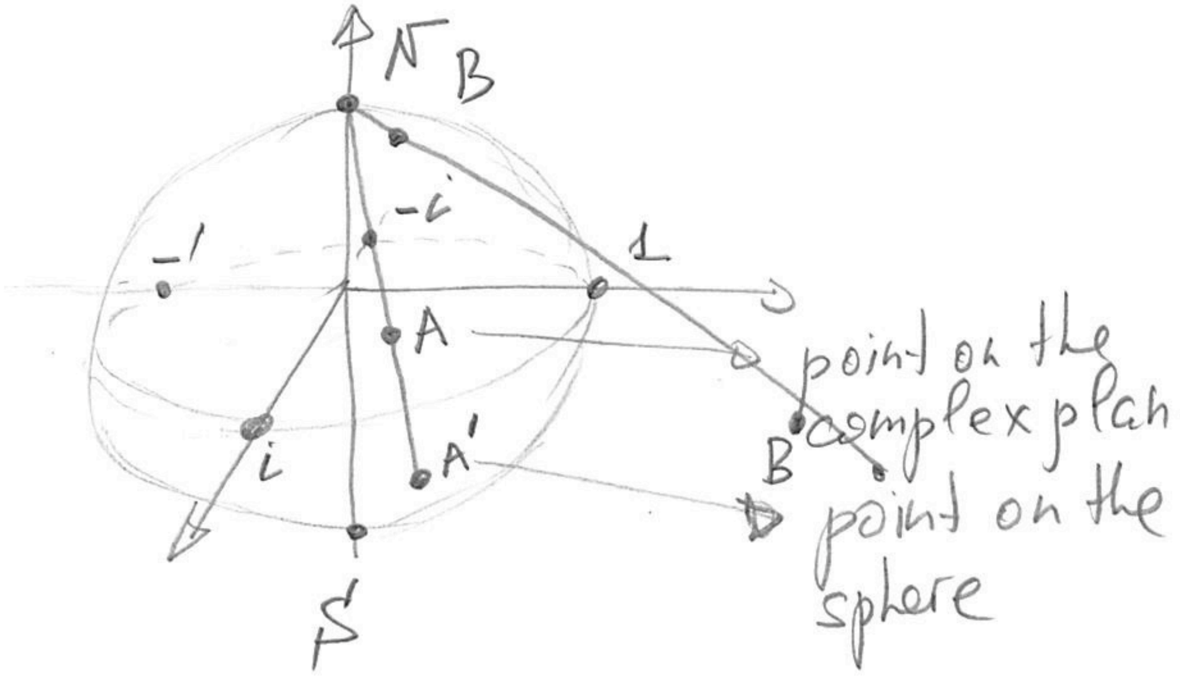
To explain this construction we use stereographic projection.

This geometrical construction was known to ancient mathematician in ancient Egypt and Greece and [Ptolemey] and artists [see Wikipedia for pictures]

To explain the idea let start with 2d version



Now we do the same geometric construction for the sphere embedded in 3D
$x^2+y^2+z^2=1$



$N \leftrightarrow (0,0,1)$

$S \leftrightarrow (0,0,-1)$

complex plane $z=0$

Unit circle on the complex plane
$x^2+y^2=1$

points on the equatorial line are mapped to the points on the unit circle.

points in the southern hemisphere are mapped inside the unit circle on the complex plane

origin $\rightarrow$ S

Points on the northern ~~h~~hemisphere are mapped out side the unit circle
The close one moves to the northern pole on the Riemann sphere the further away from the origin is its image on the complex plane.
The point N itself has no unique image at the ordinary complex plane and is mapped to "infinity".
All other points on the sphere have 1 to 1 correspondance to points on the plane. Let us add formaly one extra point to the complex plane $C \cup \infty \equiv C_\infty$
This is an extanded complex plane, that is equivalent to the Riemann surface.
What is the advantage of this construction?
Formaly we now have defined operation like

$$\frac{z}{0} = \infty \qquad \frac{z}{\infty} = 0 \qquad \text{for } z \neq 0 \text{ or } \infty$$

$$z \times \infty = \infty$$

$$z + \infty = \infty$$

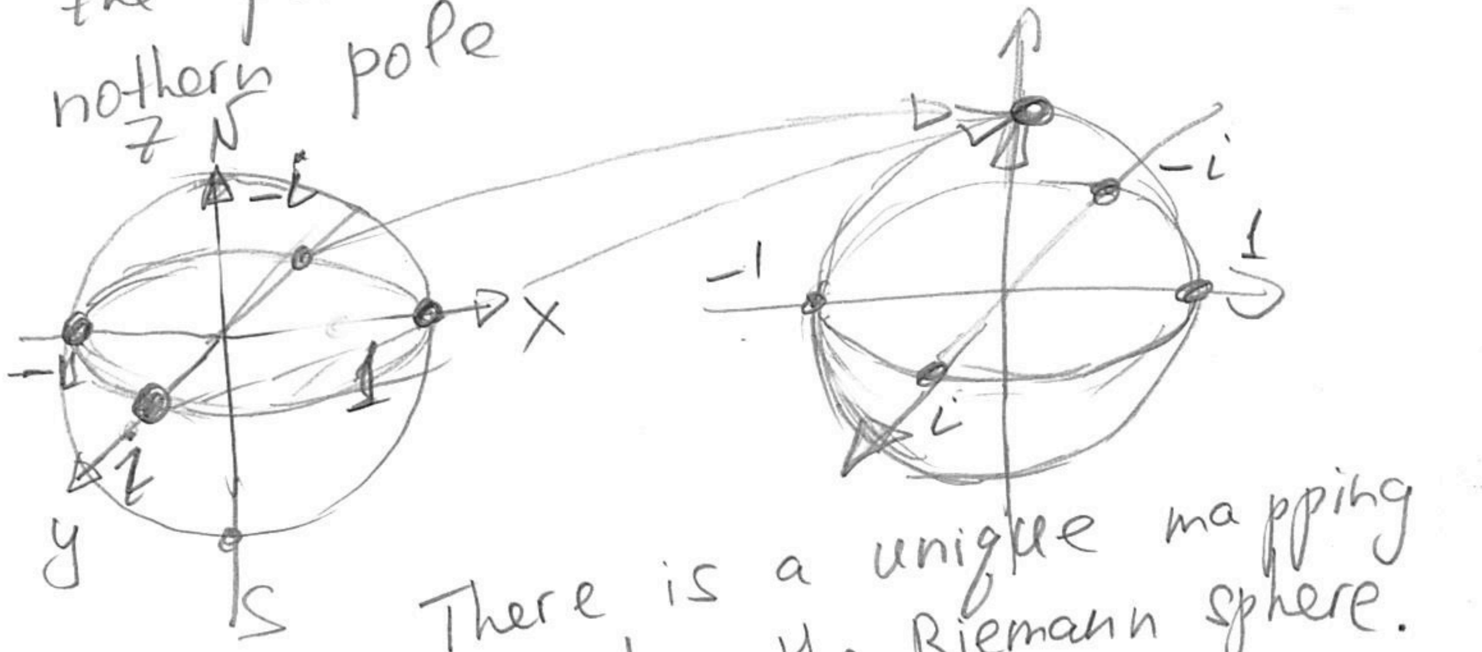
Consider now the function

$$f(z) = \frac{g(z)}{h(z)}$$ where $g(z), h(z)$ are polynomials

example

$$f(z) = \frac{z-3}{(z+i)(z-i)}$$

On the complex plane this was analytic function, except the point $z = \pm i$
On the extended complex plane [Riemann sphere] the function $f(z)$ is analytic everywhere. As a mappin on a sphere the point $z = i$ is mapped onto the nothern pole



There is a unique mapping from $\mathbb{C}$ onto the Riemann sphere.

Consider the mapping $\left(\frac{+1}{z}\right)$

Point $z=0$ on the complex plane [corresponds to S] $\xrightarrow{\text{mapped to}}$ N

Point $z=\infty$ on the complex plane [corresponds to N] $\xrightarrow{\text{mapped to}}$ S

Point i and $-i$ unit circle are interchanged.
The whole transformation is therefore a rotation by π with respect to the x axes [real axes]

As we see any rational function is a continous function

$$f(z)=\frac{g(z)}{h(z)}$$

on the Riemann sphere.
We can now treat the singularity at infinity in the same way, as we did for singularities of the function at any other point. Indeed, let us apply $z=1/\zeta$ transformation and study the singularity at the point $\zeta=0$.