Möbius transformations

Lecture 7 -1-

Focus on $\omega = \frac{1}{z}$

Consider a circle of the radius r centered at point $\alpha = \operatorname{Re}\alpha + i \operatorname{Im}\alpha$

$$(x - \operatorname{Re}\alpha)^2 + (y - \operatorname{Im}\alpha)^2 = r^2$$

$$\underbrace{x^2 + y^2}_{z\bar{z}} - 2\underbrace{x}_{\left(\frac{z+\bar{z}}{2}\right)} \cdot \operatorname{Re}\alpha - 2\underbrace{y}_{\frac{z-\bar{z}}{2i}} \operatorname{Im}\alpha + \operatorname{Re}\alpha^2 + \operatorname{Im}\alpha^2 = r^2$$

$z = x + iy$

$$z\bar{z} - 2\frac{(z+\bar{z})}{2}\operatorname{Re}\alpha - \frac{2(z-\bar{z})}{2i}\operatorname{Im}\alpha + \alpha\bar{\alpha} = r^2$$

$$z\bar{z} - (z+\bar{z})\operatorname{Re}\alpha + (z-\bar{z})i\operatorname{Im}\alpha + \alpha\bar{\alpha} = r^2$$

$$z\bar{z} - z(\operatorname{Re}\alpha - i\operatorname{Im}\alpha) - \bar{z}(\operatorname{Re}\alpha + i\operatorname{Im}\alpha) + \alpha\bar{\alpha} = r^2$$

$$z\bar{z} - z\bar{\alpha} - \bar{z}\alpha + \alpha\bar{\alpha} = r^2$$

$$z\bar{z} - z\bar{\alpha} - \bar{z}\alpha = r^2 - \alpha\bar{\alpha}$$

Under the mapping $z \to \frac{1}{\omega}$

$$\frac{1}{\omega\bar{\omega}} - \frac{\bar{\alpha}}{\omega} - \frac{\alpha}{\bar{\omega}} = r^2 - \alpha\bar{\alpha}$$

$$1 - \bar{\alpha}\bar{\omega} - \alpha\omega = (r^2 - \alpha\bar{\alpha})\,\omega\bar{\omega}$$

If $r^2 - \alpha\bar{\alpha} \neq 0$

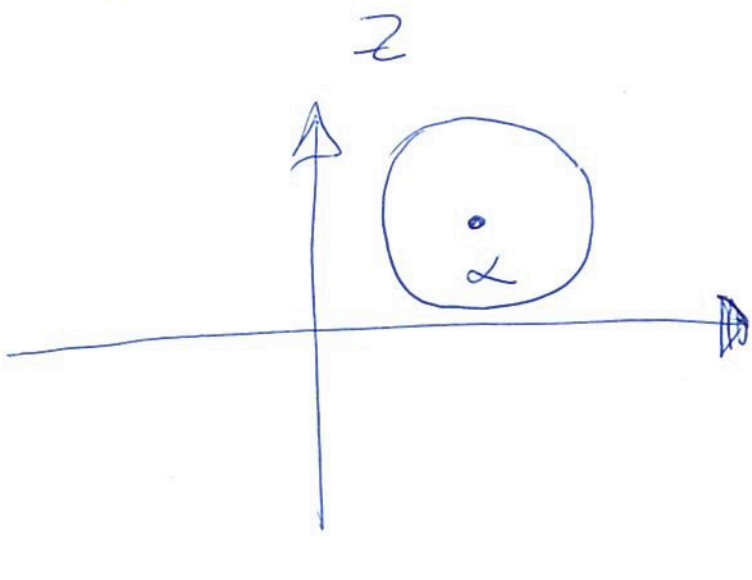
$$\omega\bar{\omega} + \frac{\alpha}{r^2-\alpha\bar{\alpha}}\,\omega + \frac{\bar{\alpha}\bar{\omega}}{r^2-\alpha\bar{\alpha}} = \frac{1}{r^2-\alpha\bar{\alpha}}$$

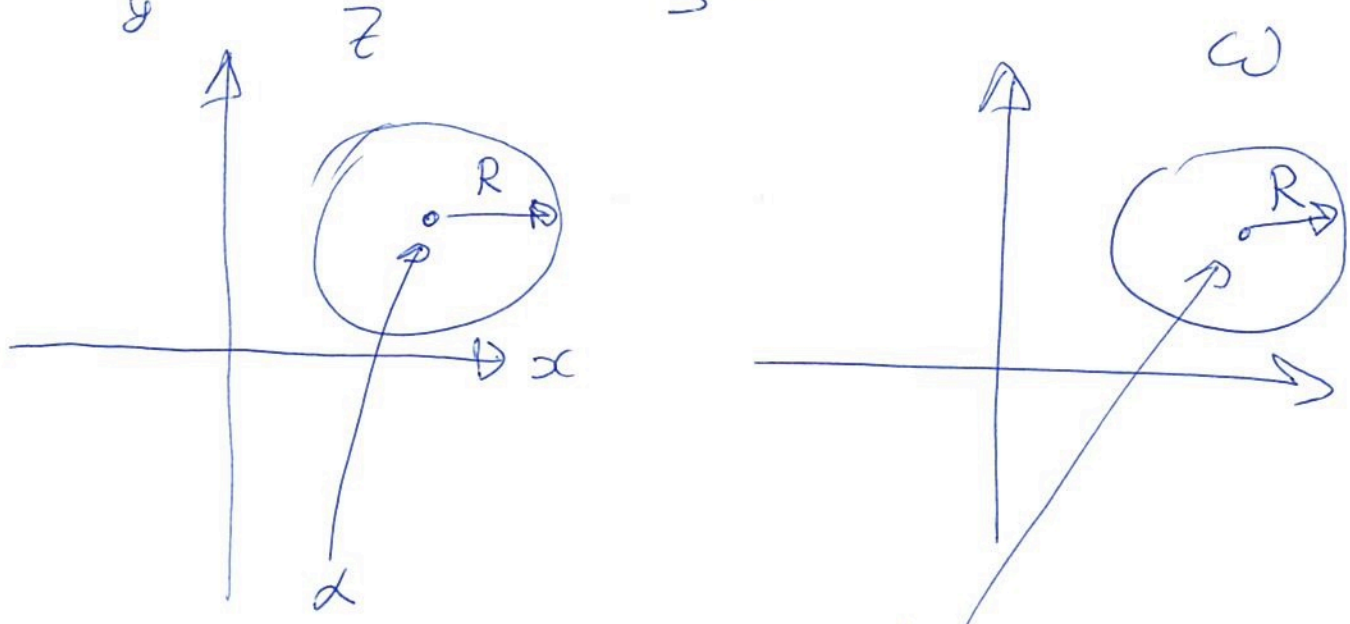
This is an equation of a circle in the complex plane ω.

If $\alpha\bar{\alpha} - r^2 = 0$

$$1 - \bar{\alpha}\bar{\omega} - \alpha\omega = 0$$

This is an equation of a straight line in ω-plane.

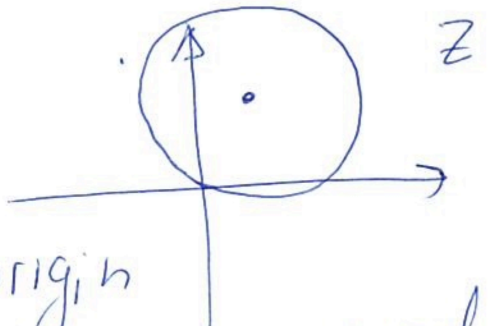




center at $\frac{\alpha}{r^2-\alpha\bar{\alpha}}$

radius

$$R^2 = \frac{1}{r^2-\alpha\bar{\alpha}} + \frac{\alpha\bar{\alpha}}{r^2-\alpha\bar{\alpha}} = \frac{1+\alpha\bar{\alpha}}{r^2-\alpha\bar{\alpha}}$$



If the circle touches the origin i.e. $r^2=\alpha\bar{\alpha}$ it is mapped onto a straight line in ω plane

Similarly, consider a straight line

$$ax+by=c$$

$$x=\frac{z+z}{2} \qquad y=\frac{z-\bar{z}}{2i}$$

$$\frac{a}{2}(z+\bar{z})+\frac{b}{2i}(z-\bar{z})=c$$

$$z\left(\frac{a}{2}+\frac{b}{2i}\right)+\bar{z}\left(\frac{a}{2}-\frac{b}{2i}\right)=c$$

$$z(a-ib)+\bar{z}(a+ib)=2c$$

Under $\omega=1/z$

$$\frac{1}{\omega}\left(\frac{ia+b}{2i}\right)+\frac{1}{\omega}\left(\frac{ia-b}{2i}\right)=c$$

$$(b+ia)\bar{\omega}+(ia-b)\omega=c\,2i\,\omega\bar{\omega}$$

$$(a-ib)\bar{\omega}+(a+ib)\omega=2c\,\omega\bar{\omega}$$

If $c\neq 0$ [line does not pass the origin]

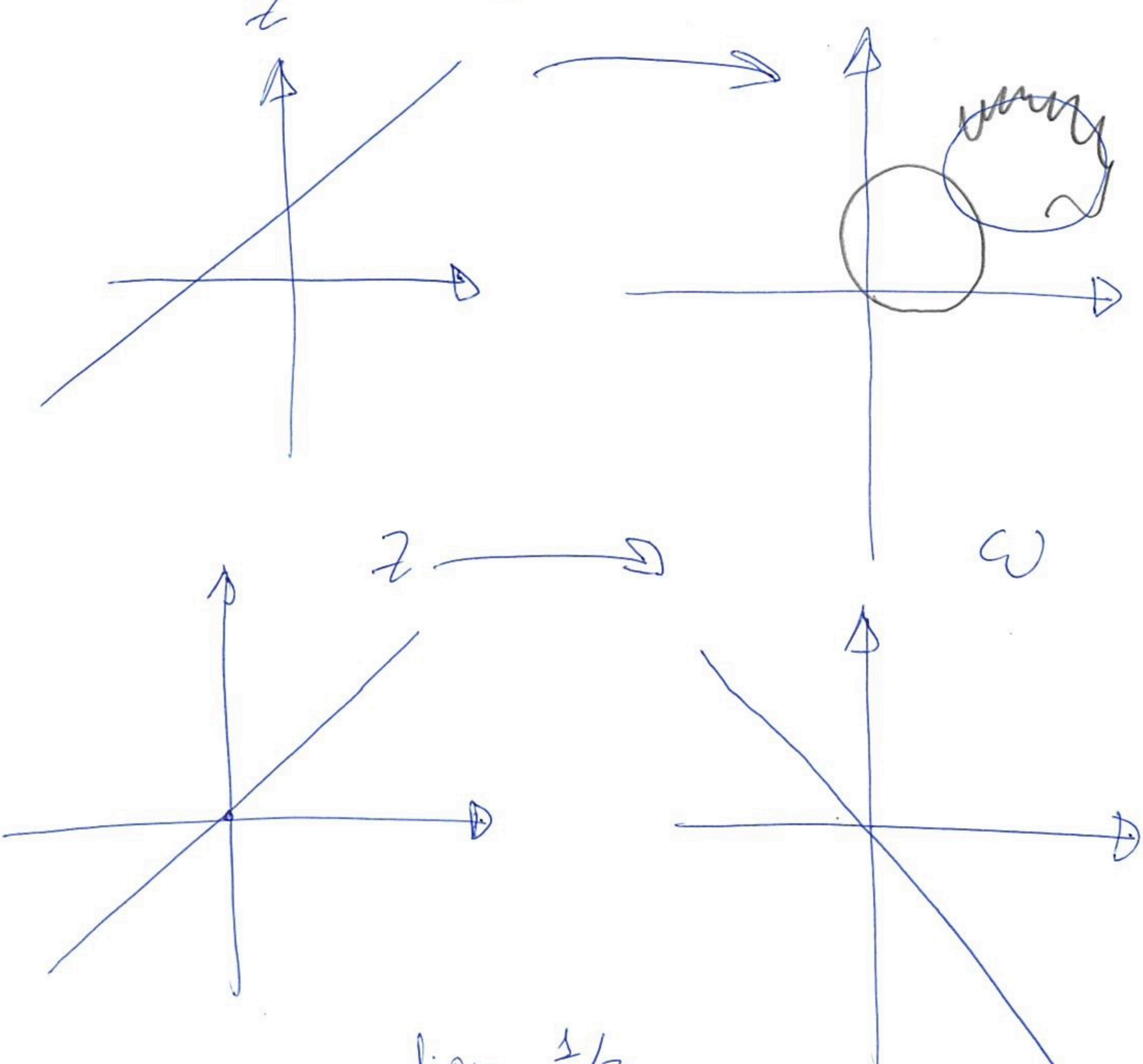
denote

$$\omega\bar{\omega}-\frac{a+ib}{2c}\omega-\frac{a-ib}{2c}\bar{\omega}=0$$

$$\omega\bar{\omega}-\bar{\alpha}\omega-\alpha\bar{\omega}=0$$

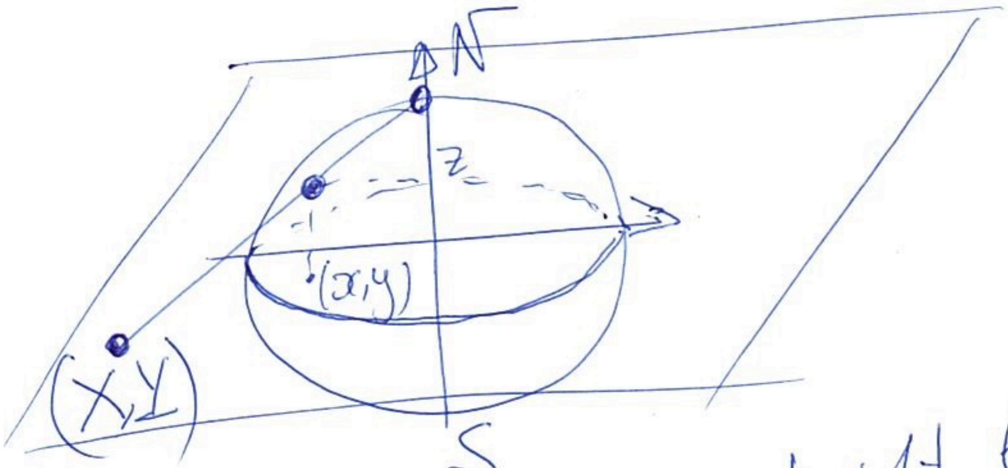
A circle of the radius R

$$R^2=\alpha\bar{\alpha}=\left(\frac{a+ib}{2c}\right)\left(\frac{a-ib}{2c}\right)=\frac{a^2+b^2}{4c^2}$$



The transformation $1/z$ maps circles and lines to circles and lines.

Note that if we look at the lines and circles on the complex plane on the Riemann sphere they are always circles.



In particular, any straight line on z plane is a circle that passes through the N-pole.

Riemann sphere

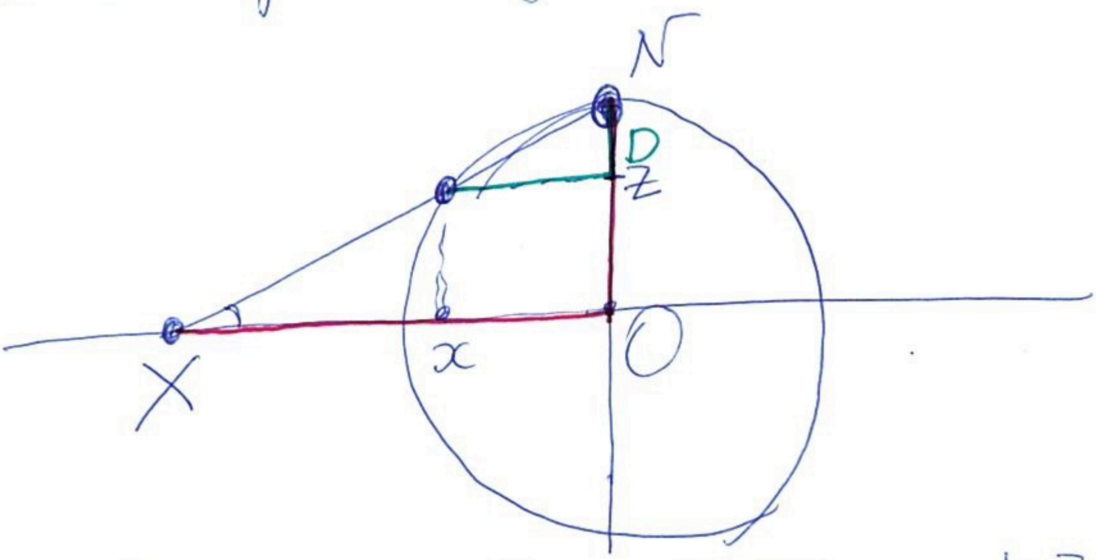
$$x^2+y^2+z^2=1$$

point (x,y,z) on the sphere is mapped on to a point $\left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ on the plane

Use capital letters for the plane

$$X=\frac{x}{1-z} \qquad Y=\frac{y}{1-z}$$

choose a point with $y=0$



$$\frac{Ox}{OX} = \cancel{\frac{1-z}{1}} \quad \frac{DN}{ON} = \frac{1-z}{1}$$

$$OX = \frac{Ox}{1-z} = \frac{x}{1-z}$$

Clear from the symmetry that that both point transform in the same way

$$X = \frac{x}{1-z} \qquad Y = \frac{y}{1-z}, \text{ where } z^2+x^2+y^2=1$$

Plane crossing the sphere $\to$ circle

$$ax+by+cz=d$$

$$z = \frac{d-ax-by}{c}$$

$$x^2+y^2+z^2=1$$

The reverse correspondance

$$(x,y,z) = \frac{1}{1+X^2+Y^2}\left(2X, 2Y, X^2+Y^2-1\right)$$

point on a sphere

If the point on a plane moves on a circle around origin

$X = R\cos\varphi$

$Y = R\sin\varphi$

$$(x,y,z) = \frac{1}{1+R^2}\left(2R\cos\varphi, 2R\sin\varphi, R^2-1\right)$$

obviously a circle at a constant z.

If the point on the plane moves on the straight line passing through the origin

$X = Y$

$$(x,y,z) = \frac{1}{1+2X^2}\left(2X, 2X, 2X^2-1\right)$$

This lies in the plane x, y. in this plane unit sphere crossed by a plane $x=y$

$$x^2+y^2+z^2=1$$

on a plane
$x=y$ $\quad 2x^2+z^2=1$

Check $\quad 2\cdot\left(\frac{2\cancel{x}}{1+2x^2}\right)^2+\left(\frac{2x^2-1}{1+2x^2}\right)^2=$

$$=\frac{1}{(1+2x^2)^2}\left[2\cdot 4x^2+(4x^4-4x^2+1)\right]$$

$$=\frac{1}{(1+2x^2)^2}\left[4x^2+4x^2+1\right]=$$

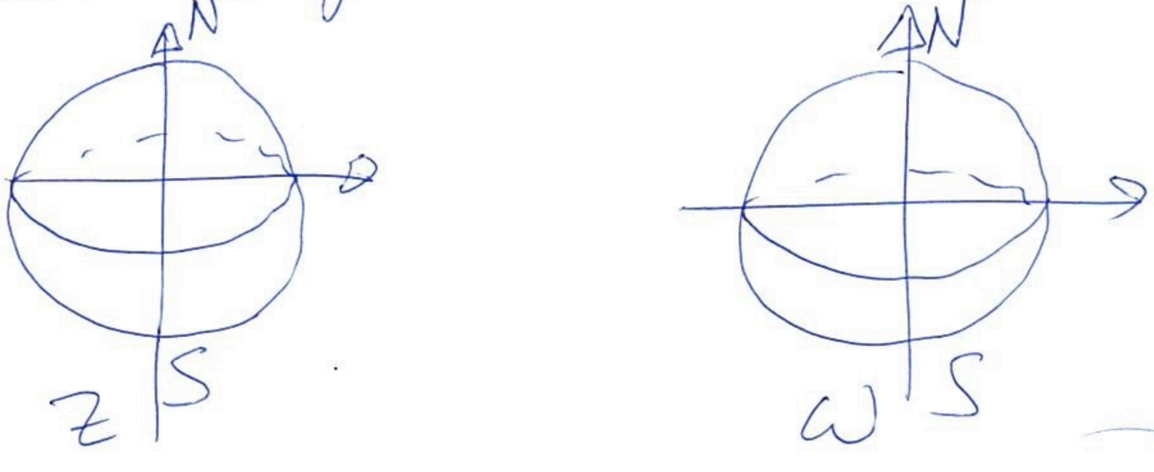
$$=\frac{(1+2x^2)^2}{(1+2x^2)^2}=1 \checkmark$$

Therefore the line $X=Y$ on the complex plane draws the circle above it on the sphere. poins at ∞ map to the N. pole.

In general, the transformation

$w=\frac{az+b}{cz+d}$ corresponds to the rotation of the Rieman sphere

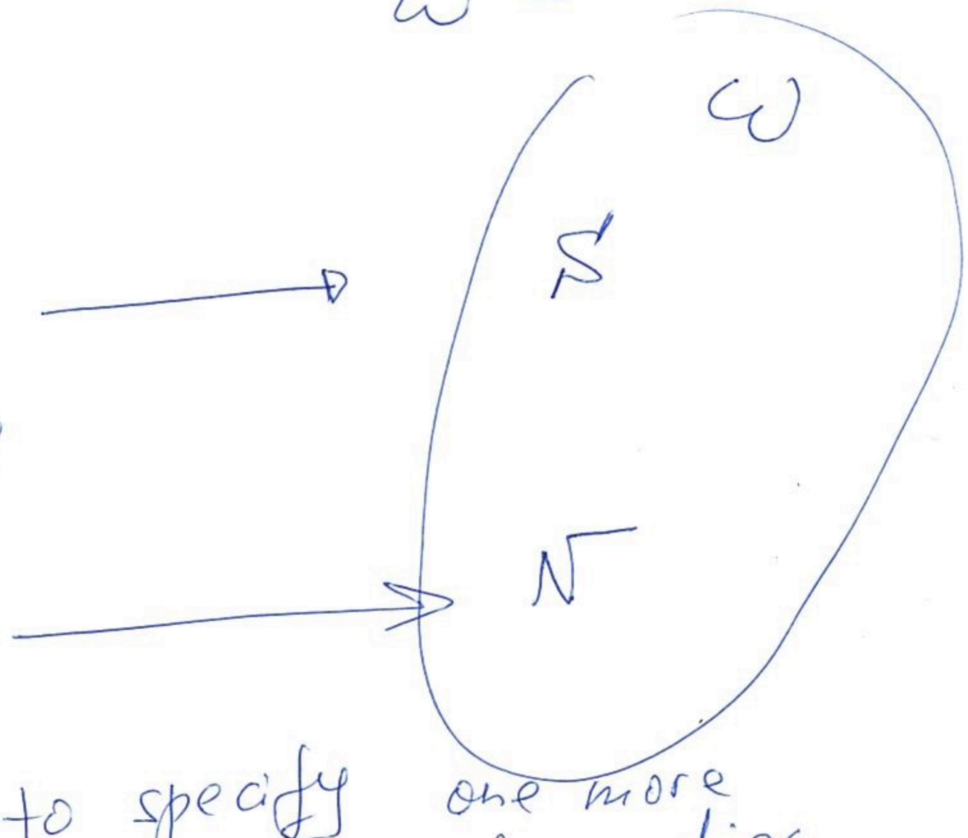
It send the points on the Rieman sphere to the Rieman sphere



point of zero

$az+b=0$

$z=-b/a$



point of pole

$cz+d=0$

$z=-d/c$

we need to specify one more point to define the transformation

$$\omega = \left(\frac{z-z_0}{z-z_\infty}\right)\left(\frac{z_1-z_\infty}{z_1-z_0}\right)$$

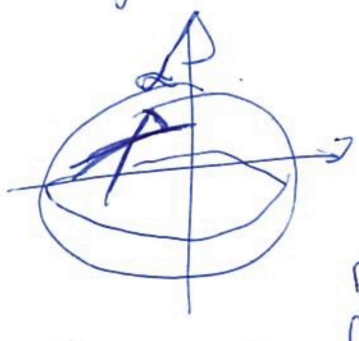
$z_0 \to 0$, i.e. S'

$z_\infty \to \infty$, ie. N

$z_1 \to 1$

- Note that the composition of Möbius transformation is again a Möbius transformation. Indeed, the composition of the rotations is a rotation.
- Since the rotation ~~of~~ a sphere preserves the angles on the sphere the mapping is conformal.

Indeed ~~any cross~~ imagine a circles on a sphere that cross at angle α.



When sphere is rotated the angle does not change and the circles remain circles.

Than, we know that this circles correspond to the circles or lines on the complex plane. Since the angle α is preserved on a sphere it is also preserved on a plane.

- Möbius transformation preserves angles globaly, between any crossing lines on a plane. It is the only global one-to-one mapping between Riemann surfaces.

- One can identify the transformation

$$\omega = \frac{az+b}{cz+d}$$ with the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ with the demand $ab-cd \neq 0$

Under this condition it is an element of a groub.

The composition of two Möbius transformations is a product of group elements

$$\omega = \frac{a_1 z + b_1}{c_1 z + d_1} \qquad \zeta = \frac{a_2 \omega + b_2}{c_2 \omega + d_2}$$

$$\omega(\zeta(z)) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$