Reminder

If $f(z) = u(x,y) + i\,v(x,y)$ is analytic than $u(x,y)$ and $v(x,y)$ are harmonic

$\partial_x^2 u + \partial_y^2 u = 0$ and $\partial_x^2 v + \partial_y^2 v = 0$

Example 1

Find harmonic function $u(x,y)$ on the complex plane that satisfies the boundary condition $u(x) = x$ on the real axis.

Define complex analyt. function

$$f(z) = z = \underbrace{x}_{u(x,y)} + i\underbrace{y}_{v(x,y)}$$

$u(x,y) = x$

$v(x,y) = y$

$\partial_x^2 x + \partial_y^2 x = 0$

$\partial_x^2 y + \partial_y^2 y = 0$

Example 2

$u(x,y) = x^2$ on the real axis

$$z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + \underbrace{2xy}_{v(x,y)}\, i$$

on the real axis

$\mathrm{Re}\, z^2 = x^2$

$u(x,y) = x^2 - y^2 = \mathrm{Re}\, z^2$

Check that the function is harmonic

$$\partial_x^2 u + \partial_y^2 u = \partial_x^2 x^2 + \partial_y^2(-y^2) = 2 - 2 = 0$$

Example 3

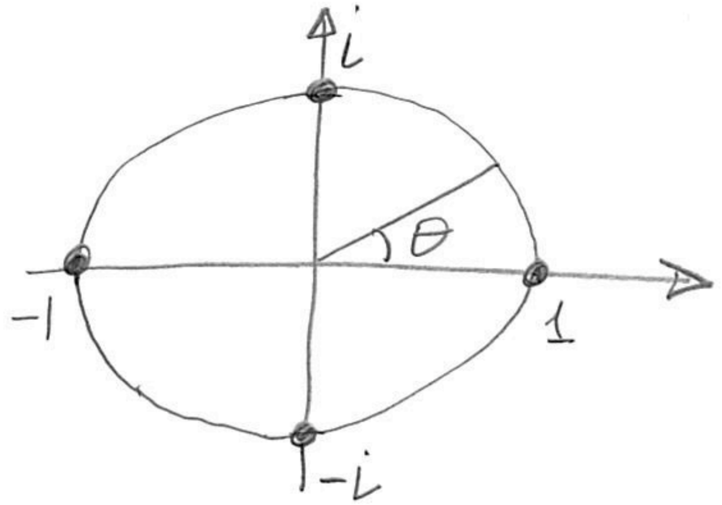
Find harmonic function that on the imaginary axis $u(0,y) = y$

Define $f(z) = -iz = \underbrace{y}_{u} - ix$

$u(x,y) = \mathrm{Re}(-iz) = y$

$$\partial_x^2 u + \partial_y^2 u = \partial_x^2 y + \partial_y^2 y = 0$$

Example 4



–||– on the unit circle

$u(\theta) = \cos 3\theta$

Define $f(z) = z^3 \Big|_{\substack{\text{on the}\\ |z|=1}} = \left(e^{i\theta}\right)^3 = e^{3i\theta}$

$\operatorname{Re} f(z) = \cos 3\theta = u(\theta)$

$u(z) = \operatorname{Re} z^3 = \operatorname{Re}(x+iy)^3 = x^3 - 3xy^2$

$v(z) = \operatorname{Im}(x+iy)^3 = 3x^2y - y^3$

$\partial_x^2 u + \partial_y^2 u = 6x - 6x = 0$

$\partial_x^2 v + \partial_y^2 v = 6y - 6y = 0$

Example 5

Find harmonic function $u(x,y)$ on the upper half plane that satisfies the boundary conditions

$$u(x,0) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases}$$

define $\theta = \operatorname{Arg} z$

$$u(x,y) = \frac{1}{\pi}\theta$$

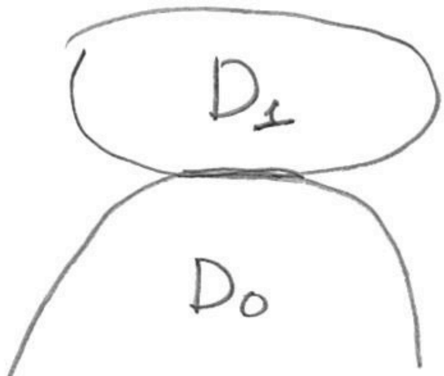
$$\log z = \ln|z| + i\theta$$

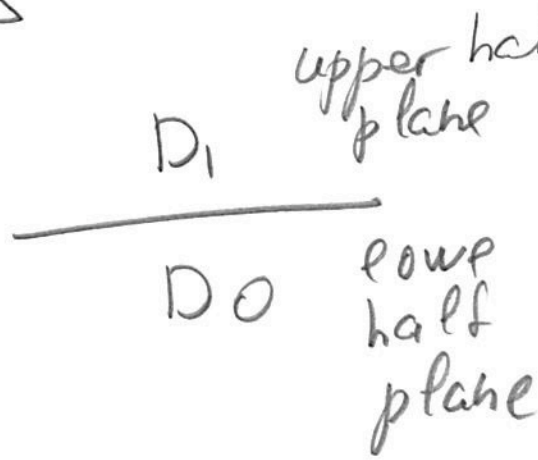
$$u(x,y) = \operatorname{Im}\left(\frac{1}{\pi}\log z\right)$$

choose the branch cut in the lower half plane

In all the examples above we actually used the idea of analytic continuation.

Suppose there are two domains with a common boundary γ





and there are functions analytic f_1 in D_1 and f_0 in D_0

If $f_1(z) = f_0(z)$ for $z \in \gamma$ and these functions are continous, than

$$f(z) = \begin{cases} f_0(z), & z \in D_0 \\ f_0(z) = f_1(z), & z \in \gamma \\ f_1(z), & z \in D_1 \end{cases}$$

is an analytic continuation of f_0 to D_1 [or of f_1 to D_0].

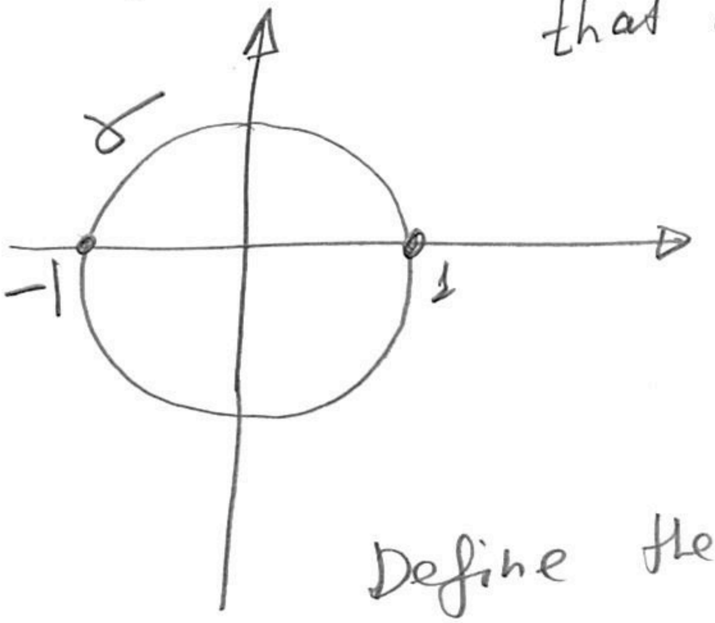
Example of analytic continuation

Consider an infinite series

$$f_0(z) = 1 + z + z^2 + \dots \; z^n + \dots$$

that converges for

$$|z| < 1$$



Define the function

$$f(z) = \frac{1}{1-z}$$

It it is equal to $f_0(z)$ on γ (except $z=1$)

therefore $f(z)$ is analytic continuation of $f_0(z)$ on the entire $\mathbb{C}$.

Les trivial example

Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(0) = \sum_{n=1}^{\infty} \frac{1}{n^0} = 1+1+1+\ldots = -\frac{1}{2}$$

Obviously this looks strange

consider

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z} + 2\sum_{n=1}^{\infty}\frac{1}{(2n)^z} =$$

Converges for $\operatorname{Re} z > 1$

even terms have a wrong sign

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z} + \frac{2}{2^z} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^z}}_{\zeta(z)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z} + 2^{1-z}\zeta(z)$$

$$\zeta(z)\left(1-2^{1-z}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$$

$$\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$$

This converges for $\operatorname{Re} z > 0$
(because of the sign alternation)

limit of $\lim_{z \to 0} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z} = 1/2$

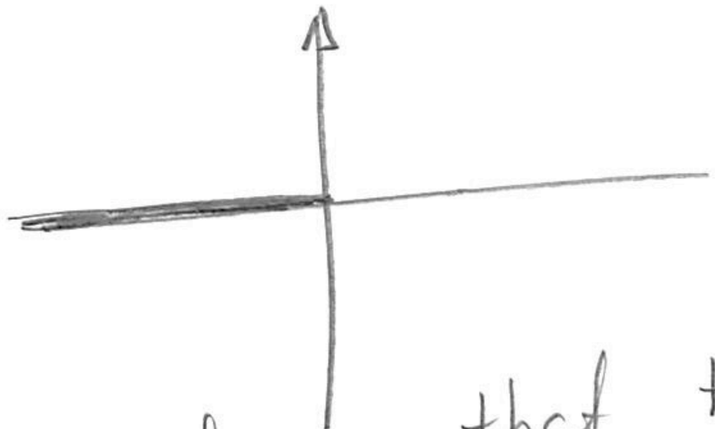
$$\zeta(z=0) = \frac{1}{1-2^1} = -\frac{1}{2}$$

We have analyticaly continued
the function
$\zeta(s)$ from the values $\operatorname{Re} s > 1$
where the series converges to
the region where it does not
There we actualy understand the
function as analytic continuation

Finally, let us discuss the analytic continuation for multivalued functions

Consider for example $f(z) = \sqrt{z}$

Function $\sqrt{z}$ has a cut, that we can choose along negative real axis



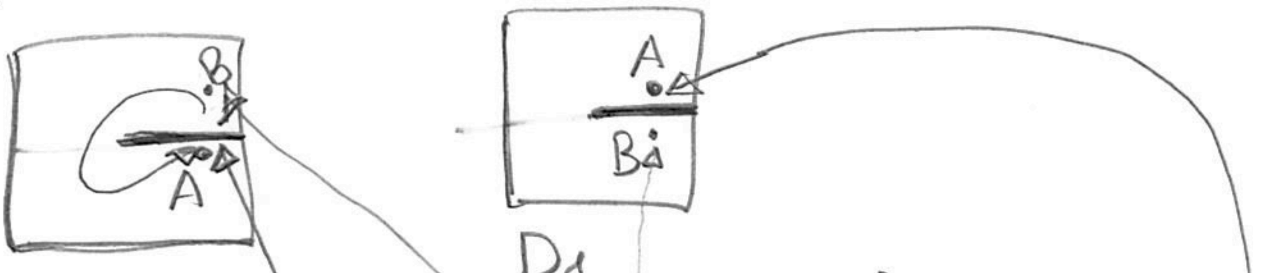
We know that the function has two branches

$$z = |z| e^{i\theta}$$

$$\sqrt{z} = |z|^{1/2} e^{i\frac{\theta}{2}}$$

$$\sqrt{z} = |z|^{1/2} e^{i\frac{\theta}{2} + \pi i}$$

Let us make two copies of the complex plane with a cut along positive real axis and denote it



D_0 $\qquad$ D_1

$|z|^{1/2}e^{i\theta/2}$ $\qquad$ $|z|^{1/2}e^{\frac{i\theta}{2}+\pi i}$

branch # 1

notet that here this branch yields $-|z|^{1/2}$ which is the same as here

Glue this lists together along the cut

continue along the second list.

Notet that in the point B the second branch yilds $e^{\pi i+\frac{i}{2}2\pi}|z|^{1/2} = e^{\pi i\times 2} = 1$ the same as here

"θ"

and we glue the list again. What we got is a (self intersecting) 2D surface in 3D, called Riemann surface

We can not make it from paper because we ignore self intersection. But on this manifold the function

$\sqrt{z}$ is single valued.

for any point z on the Riemann surface the value $\sqrt{z}$ is uniquely defined.

Similar construction can be done for any analytic function.
For example $z^{1/n}$ has a Riemann surface that consists on n list.
For logarithm the Rieman surface has infinitely many sheets.