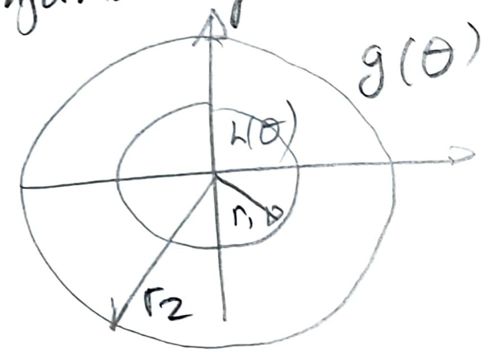
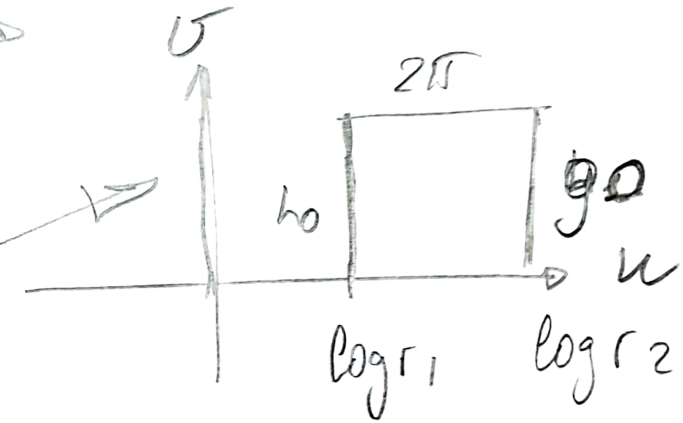
Solving Laplace eq. via conformal maps.

Examples



$z \xrightarrow{w = \ln z} w$



$\log r_1 e^{i\theta}$

$g(\theta) = g_0$

$h(\theta) = h_0$

b.c.

$f(u, 2\pi) = f(u, 0)$

$f(\ln r_1) = h_0$

$f(\ln r_2) = g_0$

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) f = 0$$

$\mathrm{Re}\, w = \mathrm{Re} \ln z = \ln |z|$

$$f(u) = h_0 + \frac{(u - \ln r_1)}{\ln(r_2) - \ln(r_1)} (g_0 - h_0)$$

$$f = h_0 + \mathrm{Re}\left(\frac{\ln z - \ln(r_1)}{\ln(r_2) - \ln(r_1)} (g_0 - h_0)\right) =$$

$$= h_0 + \frac{\ln|z| - \ln(r_1)}{\ln(r_2) - \ln(r_1)} (g_0 - h_0)$$

Laplace in spherical coordinates

check the results:

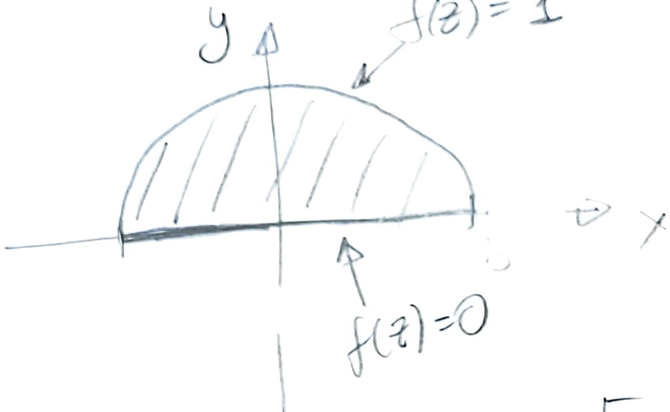
$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{d^2 f}{d\varphi^2} =$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\frac{\ln r}{\ln(r_2/r_1)}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{1}{r}\,\frac{1}{\ln r_2/r_1}\right) = \frac{1}{r}\frac{d}{dr}(1) = 0$$

z

$|z|<1$

$\operatorname{Im} z > 0$

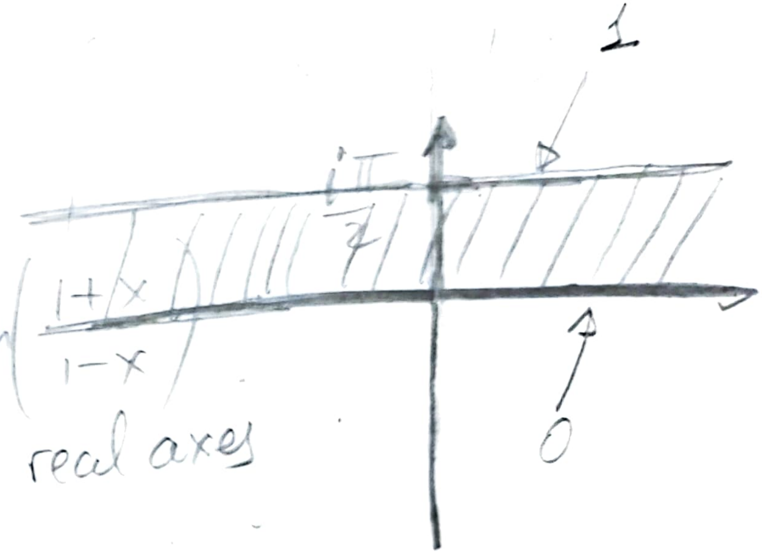


$\Delta f = 0$

ω

$$\omega = \ln \frac{1+z}{1-z}$$

$[-1, 1] \longrightarrow \ln\left(\frac{1+x}{1-x}\right)$ real axes



$$\ln \frac{1+e^{i\varphi}}{1-e^{i\varphi}} = \ln \frac{(1+e^{i\varphi})(1-e^{-i\varphi})}{(1-e^{i\varphi})(1-e^{-i\varphi})} = \ln \frac{1-e^{-i\varphi}+e^{i\varphi}-1}{1-e^{-i\varphi}-e^{i\varphi}+1} =$$

$$= \ln \frac{e^{i\varphi}-e^{-i\varphi}}{2-2\cos\varphi} = \ln 2i \frac{\sin\varphi}{2(1-\cos\varphi)} = \ln i \frac{\sin\varphi}{1-\cos\varphi} =$$

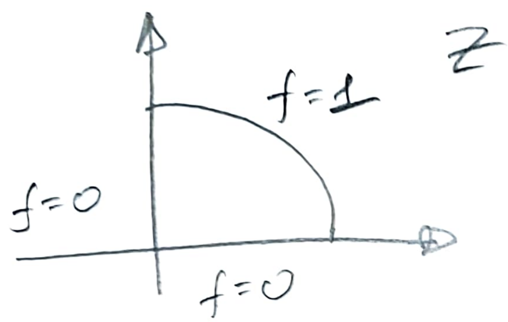
$$= \ln i + \ln\left(\frac{\sin\varphi}{1-\cos\varphi}\right) = \frac{i\pi}{2} + \ln\left(\frac{\sin\varphi}{1-\cos\varphi}\right)$$

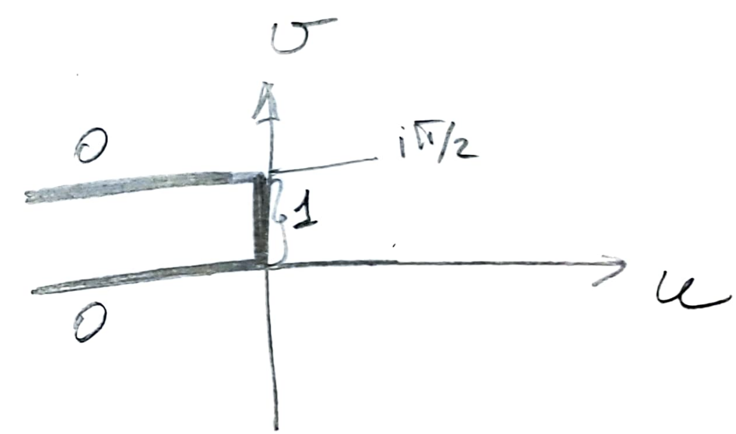
$$0 < \frac{\sin\varphi}{1-\cos\varphi} < \infty$$

$$f(\omega) = \begin{cases} 1, & \text{for } \overbrace{\mathrm{Im}\,\omega}^{v} = \pi/2 \\ 0, & \text{for } \underbrace{\mathrm{Im}\,\omega}_{v} = 0 \end{cases}$$

$$f(\omega) = \frac{v}{\pi/2} = \frac{2}{\pi} v = \frac{2}{\pi} \mathrm{Im}\,\omega =$$

$$= \frac{2}{\pi} \mathrm{Im} \ln\left(\frac{1+z}{1-z}\right) = \frac{2}{\pi} \mathrm{Arg} \frac{1+z}{1-z}$$





$w = \ln z$

$[0,1]$ $\ln x \in (-\infty, 0)$

$[i0, i1] \Rightarrow \ln ix = i\pi/2 + \ln|x|$

$\ln e^{i\varphi} = i\varphi =$

$0 < \varphi < \pi/2$

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0$$

$f(u,v) = e^{-\lambda u} \sin \lambda v = e^{-2nu} \sin 2nv$

$+\lambda^2 - \lambda^2 = 0$

$f(u, \frac{\pi}{2}) = 0$ $\lambda \frac{\pi}{2} = \pi n$

$\lambda = 2n$

$$f(u,v) = \sum_{n=1}^{\infty} e^{-2nu} \sin(2nv) A_n$$

$$f(u=0,v) = 1 = \sum_{n=1}^{\infty} \sin(2nv) A_n \qquad \text{for } v \in [0, \pi/2]$$

$$= \frac{4}{\pi} \int_0^{\pi/2} \sin(2nv) \sin(2nv) = \delta_{n,m}$$

$$A_n = \frac{4}{\pi} \int_0^{\pi/2} \sin(2nv) = \begin{cases} \frac{4}{\pi n}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$f = \sum_{\text{odd } n} e^{-2nu} \sin(2nv) \frac{4}{\pi n} = \sum_n e^{-2n \ln|z|} \sin(2n \arg(z)) \frac{4}{\pi n}$$

$$= \sum_{n \in \text{odd}} \frac{4V}{\pi n} r^{2n} \sin(2n\theta)$$

Applications in fluid dynamics

$\vec{v}(x,y)$

if $\nabla \times \vec{v} = 0 \longrightarrow \vec{v} = \nabla \phi = \left(\frac{\partial}{\partial x}\phi, \frac{\partial}{\partial y}\phi\right)$

$\mathrm{div}\,\vec{v} = \Delta \phi = 0$

harmonic function

if $\mathrm{div}\,\vec{v} = 0 \quad \vec{v}_x = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$

$$\mathrm{div}\,\vec{v} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

$\triangle$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

equivivalent to Cauchy-Riemann eq. for

$\phi = \mathrm{Re}\,F$

$\psi = \mathrm{Im}\,F$

complex potential

$$F = \phi + i\psi$$

$$\frac{dF}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial y} = v_x - i v_y$$

harmonic conjugates

flow around a 2d circle

$v(r\to\infty) = U\hat{x} = \vec{U}$

find $v(x,y)$

$\Phi(x,y) = Ux$

$\hat{v}_n(r=R)=0$

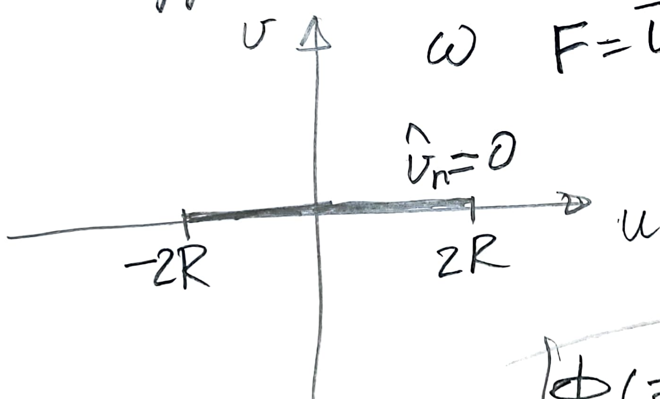
boundary conditions on a circle

conformal mapping $\omega = z + \frac{R^2}{z}$

ω $F = \vec{U}\cdot\omega$

$v(\omega\to\infty) = \vec{U}$

ω



$\Phi(\omega) = Re(\vec{U}\cdot\omega)$

$\nabla_\omega \Phi = U\,Re(1, i) = \vec{U}$

$$\boxed{\Phi(z) = Re\left(\vec{U}\left(z+\frac{R^2}{z}\right)\right)}$$

$$U = \frac{d}{dz}\Phi(z) = Re\left(\vec{U}\left(1 - \frac{R^2}{z^2}\right)\right) =$$

$$= Re\left(\vec{U}\left(1 - \frac{R^2}{r^2 e^{2i\varphi}}\right)\right) = \vec{U}\left(1 - \frac{R^2}{r^2}\cos 2\varphi\right)$$

$$\Psi(\omega) = Im(F) = Im(\vec{U}\omega)$$

$$\Psi(z) = Im\left(\vec{U}\left(z + \frac{R^2}{z}\right)\right) = Im\left(\vec{U}\left(re^{i\varphi} + \frac{R^2}{r}e^{-i\varphi}\right)\right)$$

for $z = R$

$$\Psi(R) = 2R\, Im(\vec{U}\cos\varphi) = 0$$ constant along the circle

therefore the normal component of the velocity $\hat{U}_n(r=R, \varphi) = 0$